Home Assignment 1

Problem 1.1. Show that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are $\epsilon$-isometries, then $g \circ f$ is $2\epsilon$-isometry, i.e., $\text{dis}(g \circ f) \leq 2\epsilon$.

Problem 1.2. Show that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are Lipschitz maps, then $g \circ f$ is Lipschitz and $\text{dil}(g \circ f) \leq \text{dil} f \cdot \text{dil} g$.

Problem 1.3. Let $X$ be a complete metric space and let $f : X \rightarrow X$ be a contraction (i.e., a Lipschitz map with $0 < C < 1$). Prove that there exists a unique point $x$ such that $f(x) = x$.

Hint: obtain $x$ as the limit of a sequence starting with an arbitrary $x_0$ and $x_{n+1} = f(x_n)$.

Problem 1.4. Let $(X, d)$ be a metric space and let $d'$ be the length metric induced by $d$. Denote by $d''$ the length metric induced in turn by $d'$. Show that $d' = d''$ (i.e., induction of a length metric is an idempotent operation).

Problem 1.5. Let $x : U \subset \mathbb{R}^2 \rightarrow X$ be a parametrization of the surface $X$. Show that the first fundamental form of $X$ is positive definite if and only if the parametrization is regular (i.e., at every point the coordinate vectors $x_1$ and $x_2$ span a plane).

Reminder: a quadratic form $u^T G v$ is called positive definite (denoted as $G \succ 0$) if for every $u \neq 0$, $u^T Gu > 0$.

Problem 1.6. Show two surfaces with identical first fundamental forms yet different second fundamental forms.

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Problem 1.8. Show that at $n$-th iteration of farthest point sampling, the algorithm produces an $r_n$-separated $r_n$-covering of $X$, where

$$r_n = \min_{i=1, \ldots, n} d(x, x_i),$$

and $\{x_1, \ldots, x_n\}$ are the points selected by the farthest point sampling.
Problem 1.9.  \textbf{(Schwarz lantern)} The Schwarz lantern is a triangular mesh approximating the unit cylinder $(\cos u, \sin u, v), (u, v) \in [0, 2\pi] \times [0, 1]$ and is constructed the following way: The rectangle $[0, 2\pi] \times [0, 1]$ is sampled at $n \times m$ points, where each even row of points is shifted by $\frac{\pi}{n}$ in $u$. The surface is rolled into a cylinder and triangulated as shown in Figure 1.

1. Express the area of the Schwarz lantern as a function of $m$ and $n$. What are conditions on $n, m$ to have the discrete area converge to the continuous one?

2. Express the maximum angular difference between the normal to the unit cylinder and the normal to the Schwarz lantern as a function of $m$ and $n$. What are conditions on $n, m$ to have this difference converge to zero?

3. Express the Hausdorff distance between the Schwarz lantern and a unit cylinder as a function of $m$ and $n$. What are conditions on $n, m$ to have this distance converge to zero?

\textbf{Reminder:} given two closed sets $X, Y \subset \mathbb{R}^3$, the \textit{Hausdorff distance} between them is given by

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d_{\mathbb{R}^3}(x, Y), \sup_{y \in Y} d_{\mathbb{R}^3}(y, X) \right\},$$

where $d_{\mathbb{R}^3}(x, Y) = \inf_{y \in Y} d_{\mathbb{R}^3}(x, y)$ is the point-to-set distance.