Home Assignment 1

Problem 1.1. Show that if $f : X \to Y$ and $g : Y \to Z$ are ϵ-isometries, then $g \circ f$ is 2ϵ-isometry, i.e., $\mathrm{dis}(g \circ f) \le 2\epsilon$.

Problem 1.2. Show that if $f : X \to Y$ and $g : Y \to Z$ are Lipschitz maps, then $g \circ f$ is Lipschitz and $\mathrm{dil}(g \circ f) \le \mathrm{dil}\, f \cdot \mathrm{dil}\, g$.

Problem 1.3. Let X be a complete metric space and let $f : X \to X$ be a contraction (i.e., a Lipschitz map with $0 < C < 1$). Prove that there exists a unique point x such that $f(x) = x$.

Hint: obtain x as the limit of a sequence starting with an arbitrary x_0 and $x_{n+1} = f(x_n)$.

Problem 1.4. Let (X, d) be a metric space and let d' be the length metric induced by d. Denote by d'' the length metric induced in turn by d'. Show that $d' = d''$ (i.e., induction of a length metric is an idempotent operation).

Problem 1.5. Let $x : U \subset \mathbb{R}^2 \to X$ be a parametrization of the surface X. Show that the first fundamental form of X is positive definite if and only if the parametrization is regular (i.e., at every point the coordinate vectors x_1 and x_2 span a plane).

Reminder: a quadratic form $u^{\mathrm{T}} G v$ is called *positive definite* (denoted as $G \succ 0$) if for every $u \ne 0$, $u^{\mathrm{T}} G u > 0$.

Problem 1.6. Show two surfaces with identical first fundamental forms yet different second fundamental forms.

Problem 1.7. Show two surfaces with identical second fundamental forms yet different first fundamental forms.

Problem 1.8. Show that at n-th iteration of farthest point sampling, the algorithm produces an r_n-separated r_n-covering of X, where

$$r_n = \min_{i=1,\ldots,n} d(x, x_i),$$

and $\{x_1, \ldots, x_n\}$ are the points selected by the farthest point sampling.

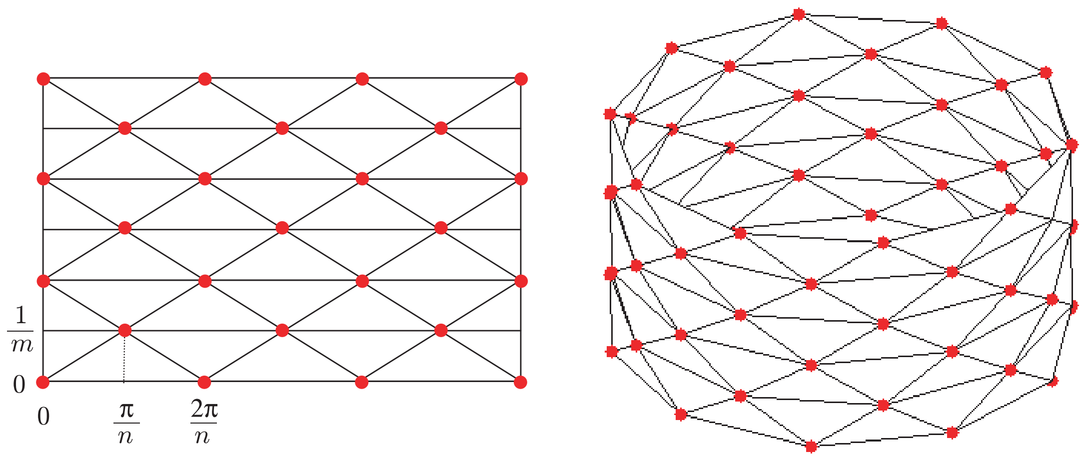

Figure 1: Schwarz lantern.

Problem 1.9. **(Schwarz lantern)** The Schwarz lantern is a triangular mesh approximating the unit cylinder $(\cos u, \sin u, v)$, $(u, v) \in [0, 2\pi] \times [0, 1]$ and is constructed the following way: The rectangle $[0, 2\pi] \times [0, 1]$ is sampled at $n \times m$ points, where each even row of points is shifted by $\frac{\pi}{n}$ in u. The surface is rolled into a cylinder and triangulated as shown in Figure 1.

1. Express the area of the Schwarz lantern as a function of m and n. What are conditions on n, m to have the discrete area converge to the continuous one?

2. Express the maximum angular difference between the normal to the unit cylinder and the normal to the Schwarz lantern as a function of m and n. What are conditions on n, m to have this difference converge to zero?

3. Express the Hausdorff distance between the Schwarz lantern and a unit cylinder as a function of m and n. What are conditions on n, m to have this distance converge to zero?

Reminder: given two closed sets $X, Y \subset \mathbb{R}^3$, the *Hausdorff distance* between them is given by

$$d_{\mathrm{H}}(X, Y) = \max \left\{ \sup_{x \in X} d_{\mathbb{R}^3}(x, Y), \sup_{y \in Y} d_{\mathbb{R}^3}(y, X) \right\},$$

where $d_{\mathbb{R}^3}(x, Y) = \inf_{y \in Y} d_{\mathbb{R}^3}(x, y)$ is the point-to-set distance.