Problem 1.1. Show that the fast marching update scheme presented in the lecture is numerically stable.

Hint: Express the derivative of $d_3$ with respect to $d_1$ and $d_2$ and show that its magnitude is always bounded by 1.

Problem 1.2. The fast marching update was derived based on the planar wavefront model expressed by the distance function $d(x) = n^T x + p$, where $p$ is the source offset and $n$ is the propagation direction. A different model is that of a spherical wavefront expressed by $d(x) = \|x - x_0\|$, where $x_0$ is a source point. Derive an update scheme for the spherical wavefront explicitly expressing

1. the update equation;
2. consistency conditions;
3. monotonicity conditions;
4. error growth.

Explain whether this scheme always guarantees to produce a consistent update. Is the scheme numerically stable?

Problem 1.3. Let us be given a surface $X$ with a Riemannian metric and let $v : X \to (0, \infty)$ be a positive scalar velocity field attached to it. We define the length structure as

$$L(\Gamma) = \int_0^1 \frac{1}{v(\Gamma(t))} \sqrt{\langle \dot{\Gamma}(t), \dot{\Gamma}(t) \rangle_{\Gamma(t)}} dt,$$

which can be thought of as the time a wave takes to propagate along the trajectory $\Gamma$. Show that the wavefront propagation time (i.e., the distance map associated with $L(\Gamma)$) is given by the viscosity solution of

$$\|\nabla_X d\| = \frac{1}{v}.$$

Hint: Express the non-homogenous length structure $L(\Gamma)$ as a homogenous length structure resulted from a point-wise scaled Riemannian metric.
Problem 1.4. Let us be given two sets \( \{x_i\} \) and \( \{y_i\} \) of \( n \) corresponding points in \( \mathbb{R}^3 \). Give a closed form expression for the rigid transformation \( R, t \) such that \( \{z_i = Ry_i + t\} \) minimizes
\[
d^2 = \sum_{i=1}^{n} \|x_i - y'_i\|^2.
\]

Problem 1.5. Write in matrix form the gradient w.r.t. \( Z \) of \( \text{tr}(Z^TB(Z)Z) \), where
\[
b_{ij}(Z) = \begin{cases} -d_X(x_i, x_j)d_{ij}^{-1}(Z) & i \neq j, d_{ij}(Z) \neq 0 \\ 0 & i \neq j, d_{ij}(Z) = 0 \\ -\sum_{k \neq i} b_{ij}(Z) & i = j \end{cases}
\]

Problem 1.6. How the degrees of freedom in the definition of the solution of the LS-MDS problem are manifested in the form of the matrices \( V \) and \( B(Z) \)?

Problem 1.7. Show that the constrained minimization problem
\[
\min_{Z, \tau} \tau \text{ s.t. } \left\{ \begin{array}{l} d_{ij}(Z) - d_X(x_i, x_j) \leq \tau \\ -d_{ij}(Z) + d_X(x_i, x_j) \leq \tau \end{array} \right.
\]
is equivalent to
\[
\min_{Z} \max_{i,j} |d_{ij}(Z) - d_X(x_i, x_j)|.
\]

Problem 1.8. Vector extrapolation involves finding the least-squares solution of the overdetermined linear system
\[
A\gamma = 0 \text{ s.t. } 1^T\gamma = 1, \tag{1}
\]
where \( A \) in an \( N \times (K + 1) \) matrix, \( N \gg K + 1 \). Assume that a QR-decomposition of \( A = QR \) is given, where \( Q \) in an \( N \times (K + 1) \) orthonormal matrix (i.e., \( Q^TQ = I \)), and \( R \) is an \( (K + 1) \times (K + 1) \) upper triangular matrix (i.e., \( r_{ij} = 0 \) for \( i > j \)). Show that solving (1) is equivalent to solving \( R^TR\eta = 1 \) and setting \( \gamma = \eta/(1^T\eta) \).