Home Assignment 2

Problem 1.1. Show that the fast marching update scheme presented in the lecture is numerically stable.

Hint: Express the derivative of d_3 with respect to d_1 and d_2 and show that its magnitude is always bounded by 1.

Problem 1.2. The fast marching update was derived based on the *planar* wavefront model expressed by the distance function $d(x) = n^{\mathrm{T}}x + p$, where p is the source offset and n is the propagation direction. A different model is that of a *spherical* wavefront expressed by $d(x) = \|x - x_0\|$, where x_0 is a source point. Derive an update scheme for the spherical wavefront explicitly expressing

1. the update equation;
2. consistency conditions;
3. monotonicity conditions;
4. error growth.

Explain whether this scheme always guarantees to produce a consistent update. Is the scheme numerically stable?

Problem 1.3. Let us be given a surface X with a Riemannian metric and let $v : X \to (0, \infty)$ be a positive scalar *velocity field* attached to it. We define the length structure as

$$L(\Gamma) \;=\; \int_0^1 \frac{1}{v(\Gamma(t))} \sqrt{\langle \dot{\Gamma}(t), \dot{\Gamma}(t) \rangle_{\Gamma(t)}}\, dt,$$

which can be thought of as the time a wave takes to propagate along the trajectory Γ. Show that the wavefront propagation time (i.e., the distance map associated with $L(\Gamma)$) is given by the viscosity solution of

$$\|\nabla_X d\| \;=\; \frac{1}{v}.$$

Hint: Express the non-homogenous length structure $L(\Gamma)$ as a homogenous length structure resulted from a point-wise scaled Riemannian metric.

Problem 1.4. Let us be given two sets $\{x_i\}$ and $\{y_i\}$ of n corresponding points in $\mathbb{R}^3$. Give a closed form expression for the rigid transformation R, t such that $\{z_i = Ry_i + t\}$ minimizes

$$d^2 = \sum_{i=1}^{n} \|x_i - y_i'\|^2.$$

Problem 1.5. Write in matrix form the gradient w.r.t. Z of $\mathrm{tr}(Z^{\mathrm{T}}B(Z)Z)$, where

$$b_{ij}(Z) = \begin{cases} -d_X(x_i, x_j)d_{ij}^{-1}(Z) & i \neq j, d_{ij}(Z) \neq 0 \\ 0 & i \neq j, d_{ij}(Z) = 0 \\ -\sum_{k \neq i} b_{ij}(Z) & i = j \end{cases}$$

Problem 1.6. How the degrees of freedom in the definition of the solution of the LS-MDS problem are manifested in the form of the matrices V and $B(Z)$?

Problem 1.7. Show that the constrained minimization problem

$$\min_{Z,\tau} \tau \text{ s.t. } \begin{cases} d_{ij}(Z) - d_X(x_i, x_j) \leq \tau \\ -d_{ij}(Z) + d_X(x_i, x_j) \leq \tau \end{cases}$$

is equivalent to

$$\min_{Z} \max_{i,j} |d_{ij}(Z) - d_X(x_i, x_j)|.$$

Problem 1.8. Vector extrapolation involves finding the least-squares solution of the overdetermined linear system

$$A\gamma = 0 \text{ s.t. } 1^{\mathrm{T}}\gamma = 1, \tag{1}$$

where A in an $N \times (K+1)$ matrix, $N \gg K+1$. Assume that a QR-decomposition of $A = QR$ is given, where Q in an $N \times (K+1)$ orthonormal matrix (i.e., $Q^{\mathrm{T}}Q = I$), and R is an $(K+1) \times (K+1)$ upper triangular matrix (i.e., $r_{ij} = 0$ for $i > j$). Show that solving (1) is equivalent to solving $R^{\mathrm{T}}R\eta = 1$ and setting $\gamma = \eta/(1^{\mathrm{T}}\eta)$.