Fast marching methods

Lecture 3
Metric discretization

Approach I: discrete metric

Discretized shape → Discrete metric → Metrication error
\[ \lim d_L \neq d_X \]

"Sampling theorem"
\[ \lim d_L = d_X \]

Approach II: consistently discretized metric

Discretized metric
Imagine a forest fire...
Forest fire

- Fire starts at a **source** $x_0$ at $t = 0$.
- Propagates with **constant velocity** $v = 1\text{m/sec}$
- Arrives at time $t(x)$ to a point $x$.
- **Fermat’s (least action) principle:**
  
  *The fire chooses the quickest path to travel.*
- Governs **refraction laws** in optics (Snell’s law) and acoustics.
- Fire **arrival time** $t(x) =$ **distance map** $d(x)$ from source.
Distance maps on surfaces

- Distance map on surface $d : X \rightarrow \mathbb{R}$
  $$d(x) = d_X(x_0, x)$$

- Mapped locally to the tangent space
  $$d : T_x \rightarrow \mathbb{R}$$

- A small step in the direction $v \in T_x$ changes the distance by
  $$d(x + v) - d(x) = D_v d(x) + O(||v||^2)$$

- $D_v$ is directional derivative in the direction $v$. 
**Intrinsic gradient**

- For some direction \( \mathbf{v}_1 \in T_x \), \( d(x + \mathbf{v}_1) = d(x) \)
- The **perpendicular direction** \( \mathbf{v}_2 \perp \mathbf{v}_1 \) is the direction of **steepest** change of the distance map.
- \( \mathbf{v}_2 \) is referred to as the **intrinsic gradient**.

*Formally, the intrinsic gradient of* function \( d : X \to \mathbb{R} \) *at a point* \( x \in X \) *is a map*

\[
\nabla_X d : T_x X \to T_x X
\]

*satisfying for any* \( \mathbf{v} \in T_x \)

\[
\langle \nabla_X d(x), \mathbf{v} \rangle = D_{\mathbf{v}}d(x)
\]
Extrinsic gradient

- Consider the distance map as a function $d : \mathbb{R}^3 \rightarrow \mathbb{R}$.
- The **extrinsic gradient** of $d$ at a point $x \in X$ is a map
  \[ \nabla_x d : X \rightarrow \mathbb{R}^3 \]
  satisfying for any direction $dx$
  \[
  \langle \nabla_x d(x), dx \rangle = \frac{d}{dt}d(x + tdx) \bigg|_{t=0}
  \]
- In the **standard Euclidean basis**
  \[
  \nabla_x d = \left( \frac{\partial d}{\partial x^1}, \frac{\partial d}{\partial x^2}, \frac{\partial d}{\partial x^3} \right)^T
  \]
- Usually called “**the gradient**” of $d$.
- What is the connection between intrinsic and extrinsic gradients?
Intrinsic and extrinsic gradients

- Intrinsic gradient = projection of extrinsic gradient on tangent plane

\[ \nabla_{Xd}(x) = P_{T_xX} \nabla_{Xd}(x) \]

- In coordinates of a parametrization \( x : U \rightarrow X \),

\[ \nabla_{Xd} = J(J^T J)^{-1} J^T \nabla_{Xd} \]

- \( J \) is the Jacobian matrix whose columns span \( T_xX \).
Eikonal equation

- Let $\Gamma$ be a minimal geodesic between $x_0$ and $x$.

- The derivative $\dot{\Gamma}(t) = \frac{d}{dt}\Gamma(t)$ is the fire front propagation direction.

- In arclength parametrization $\|\dot{\Gamma}(t)\|_2 = 1$.

- Fermat’s principle:
  $$\dot{\Gamma}(t) = \nabla_X d(\Gamma(t))$$

- Propagation direction = direction of steepest increase of $d$.

- Geodesic is perpendicular to the level sets of $d$ on $X$. 
**Eikonal equation**

- **Eikonal equation** (from Greek εικών)

\[ ||\nabla_x d||_2 = 1 \]

- **Hyperbolic** PDE with **boundary** condition

\[ d(x_0) = 0 \]

- Minimal geodesics are **characteristics**.

- Describes **propagation** of **waves** in medium.
Eikonal equation

- Let \( \Gamma \) be a minimal geodesic between \( x_0 \) and \( x \).
- The derivative \( \dot{\Gamma}(t) = \frac{d}{dt} \Gamma(t) \) is the fire front propagation direction.
- In arclength parametrization \( \|\dot{\Gamma}(t)\|_2 = 1 \).
- Fermat’s principle:
  \[
  \dot{\Gamma}(t) = \nabla_X d(\Gamma(t))
  \]
- Propagation direction = direction of steepest increase of \( d \).
- Geodesic is perpendicular to the level sets of \( d \) on \( X \).
Uniqueness of solution

- In classic PDE theory, a **solution** is a **continuous differentiable** function \( d : X \to \mathbb{R} \) satisfying

\[
\| \nabla_X d(x) \|_2 = 1 \\
\quad d(x_0) = 0
\]

- PDE theory guarantees **existence** and **uniqueness** of solution.
- Distance map is not **everywhere differentiable**.
- Solution is **not unique**!
Numerical geometry of non-rigid shapes

Fast Marching Methods
Sub- and super-derivatives (1D case)

\[ D^+ d = \{ |\alpha| \leq 1 \} \]

- **Superderivative**: the set of all slopes above the graph

\[ D^+ d(x) = \left\{ \alpha : \limsup_{h \to 0} \frac{d(x+h) - d(x) - \alpha h}{|h|} \leq 0 \right\} \]

- **Subderivative**: the set of all slopes below the graph

\[ D^- d(x) = \left\{ \beta : \liminf_{h \to 0} \frac{d(x+h) - d(x) - \beta h}{|h|} \geq 0 \right\} \]

- \( D^+ d = D^- d = \{ d' \} \) where \( d \) is differentiable.
Viscosity solution

- $d$ is a **viscosity solution** of the 1D eikonal equation if
  \[
  |\alpha| \leq 1 \quad \forall \alpha \in D^+(x) \\
  |\beta| \geq 1 \quad \forall \beta \in D^-(x)
  \]

- **Monotonicity**: viscosity solution does not have **local maxima**.

- The **largest** $d$ among all
  \[
  \{d : |d'|_{\infty} \leq 1, d(x_0) = 0\}
  \]

- **Existence** and **uniqueness** guaranteed.
Fast marching methods (FMM)

- A family of numerical methods for solving eikonal equation.
- Finds the viscosity solution = distance map.
- Simulates wavefront propagation from a source set.
- A continuous variant of Dijkstra’s algorithm.
- Consistently approximate the intrinsic metric on the surface.
Fast marching algorithm

- Initialize $d(x_0) = 0$ and mark it as black.
- Initialize $d(x) = \infty$ for other vertices and mark them as green.
- Initialize queue of red vertices $Q = \emptyset$.

Repeat

- Mark green neighbors of black vertices as red (add to $Q$).
- For each red vertex $x$
  - For each triangle sharing the vertex $x$
    - Update $x$ from the triangle.
  - Mark $x$ with minimum value of $d$ as black (remove from $Q$).

Until there are no more green vertices.

Return distance map $d(x) \approx d_X(x_0, x)$. 
Update step

Dijkstra’s update
- **Vertex** updated from adjacent **vertex** $x_1$
- Distance computed from $d(x_1)$
- Path restricted to **graph edges**

Fast marching update
- **Vertex** updated from **triangle**
- Distance computed from $d(x_1)$ and $d(x_2)$
- Path can pass on **mesh faces**
Fast marching update step

- Update from triangle
- Compute \( d_1 = d(x_1) \) and \( d_2 = d(x_2) \)
- Model wave front propagating from planar source

\[ \langle x, n \rangle + p = 0 \]

- \( n \) unit propagation direction
- \( p \) source offset
- Front hits \( x_1 \) at time \( d_1 \)
- Hits \( x_2 \) at time \( d_2 \)
- When does the front arrive to ?
Fast marching update step

- Assume w.l.o.g. \( x_1, x_2, x_3 \in \mathbb{R}^2 \) and \( x_3 = 0 \).
- \( d \) is given by the point-to-plane distance
  \[ d_B \parallel p \quad \langle x_1 p d_B \parallel p \quad \langle x_1 p \]

- Solve for parameters \( n \) and \( p \) using the point-to-plane distance
  \[
  \begin{align*}
  \langle x_1, n \rangle + p &= d_1 \\
  \langle x_2, n \rangle + p &= d_2 
  \end{align*}
  \]

- In vector notation
  \[ V^T n + p \cdot 1 = d \]
  where \( V = (x_1, x_2) \), \( d = (d_1, d_2)^T \), and \( 1 = (1, 1)^T \).

- In a non-degenerate triangle matrix \( V \) is full-rank
  \[ n = (V^T)^{-1}(d - p \cdot 1) = V^{-T}(d - p \cdot 1) \]
Fast marching update step

\[ n = V^{-T}(d - p \cdot 1) \]

- Apparently, we have **two equations** with **three variables**.
- However, \( n \) is a **unit vector**, hence \( \|n\| = 1 \).

\[
1 = n^T n = (d - p \cdot 1)^T V^{-1} V^{-T}(d - p \cdot 1) \\
= (d - p \cdot 1)^T (V^T V)^{-1}(d - p \cdot 1) \\
= p^2 \cdot 1^T Q 1 - 2p \cdot 1^T Q d + d^T Q d
\]

where \( Q = (V^T V)^{-1} \).

- Substitute and obtain a **quadratic equation**
Causality condition

- **Quadratic equation** is satisfied by both \( n \) and \(-n\).
- **Two solutions** for \( n \).
- **Causality**: front can propagate only forward in time.
- **Causality condition**

\[
0 > V \cdot n
\]
Causality condition

- Causality condition: $V^T n < 0$

*In other words*

- $n$ has to form **obtuse angles** with both **triangle edges** $(x_3, x_1), (x_3, x_2)$.

- Causality is required to obtain **consistent** approximation of the distance map.

- Smallest solution for $\| \cdot \|$ is **inconsistent** and is discarded.

- If largest solution is **consistent**, live the largest solution!
Monotonicity condition

- **Viscosity solution** has to be a monotonically increasing function.
- **Monotonicity condition**: increase when $d_1$ or $d_2$ increase.

   *In other words:*

   $$\nabla_d d_3 = \left( \frac{\partial d_3}{\partial d_1}, \frac{\partial d_3}{\partial d_2} \right)^T > 0$$

- Differentiate

   $$d_3 \cdot 1^T Q 1 - 2 d_3 \cdot 1^T Q d + d^T Q d - 1 = 0$$

   w.r.t $d = (d_1, d_2)^T$ obtaining

   $$\nabla_d d_3 = \frac{Q(d - d_3 \cdot 1)}{1^T Q(d - d_3 \cdot 1)}$$
Monotonicity condition

\[ \nabla_d d_3 = \frac{Q(d - d_3 \cdot 1)}{1^T Q(d - d_3 \cdot 1)} \]

- Substitute \( n = V^{-T}(d - d_3 \cdot 1) \)

\[ \nabla_d d_3 = \frac{QV^T n}{1^T QV^T n} \]

- Monotonicity \( \nabla_d d_3 > 0 \) satisfied when both coordinates of \( QV^T n \) have the same sign.
- \( Q = (V^TV)^{-1} \) is positive definite \( \Rightarrow \) At least one coordinate of \( QV^T n \) is negative
- Causality condition: \( V^T n < 0 \) \( \Rightarrow \) Monotonicity condition: \( QV^T n < 0 \)
Monotonicity condition

- Since $Q = (V^T V)^{-1}$ we have $Q V^T V = I$
- Rows of $Q V^T$ are orthogonal to triangle edges

**Monotonicity condition:** $Q V^T n < 0$

*Geometric interpretation:*
- $n$ must form **obtuse** angles with normals to triangle edges.

*Said differently:*
- $n$ must come **from within the triangle.**
One-sided update

- **Monotonicity condition:** update direction \( n \) must come **from within the triangle**.
- If it does not, **project** \( n \) inside the triangle.
- \( n \) will coincide with one of the edges.
- Update will reduce to **Dijkstra’s update**

\[
\begin{align*}
\frac{d}{dt} x_1 &= d_{x_1} \quad d_{x_1} &= d_{x_3} \\
\frac{d}{dt} x_2 &= d_{x_2} \quad d_{x_2} &= d_{x_3}
\end{align*}
\]

or

\[
\begin{align*}
\frac{d}{dt} x_2 &= d_{x_2} \\
\frac{d}{dt} x_1 &= d_{x_3}
\end{align*}
\]
Fast marching update

- Solve for the **quadratic equation**

- Compute **propagation direction**

\[ n(d = dY \cdot 1)^T n(d = dY \cdot 1) \]

- If **monotonicity condition** \( QV^T n < 0 \) is violated,

- Set

\[ \varphi(d(x); a1) = \varphi(d(x); a2) \]
Consistency and monotonicity encore

**Acute triangle**
All directions in the triangle satisfy \textit{consistency} and \textit{monotonicity} conditions.

**Obtuse triangle**
Some \textit{directions} in the triangle violate \textit{consistency} condition!
Fast marching on obtuse meshes

- Inconsistent solution if the mesh contains obtuse triangles
- Remeshing is costly
- Solution: split obtuse triangles by adding virtual connections to non-adjacent vertices
- Done as a pre-processing step in $O(N)$
Mesh “unfolding”

- Virtual connection splits obtuse angle into two acute ones
MATLAB® intermezzo

Fast marching
Eikonal equation on parametric surfaces

- **Parametrization** $x : U \rightarrow \mathbb{R}^3$ of $X$ over $U \subset \mathbb{R}^2$.
- Compute **distance map** $d : U \rightarrow \mathbb{R}$, $d(u) = d_X(x(u_0), x(u))$ from source $u_0 \in U$.

- **Chain rule**
  \[
  \frac{\partial d}{\partial u^i} = \frac{\partial d}{\partial x^1} \frac{\partial x^1}{\partial u^i} + \frac{\partial d}{\partial x^2} \frac{\partial x^2}{\partial u^i} + \frac{\partial d}{\partial x^3} \frac{\partial x^3}{\partial u^i}
  \]

- **Extrinsic gradient** in parametrization coordinates
  \[\nabla_{u} d = J^T \nabla_{x} d\]

- **Intrinsic gradient** in parametrization coordinates
  \[\nabla_{x} d = J(J^T J)^{-1} J^T \nabla_{x} d = J G^{-1} \nabla_{u} d\]
Eikonal equation on parametric surfaces

**Eikonal equation** in parametrization coordinates

\[
1 = \| \nabla_{Xd} \|^2 = \nabla_{Xd}^{T} \nabla_{Xd} \\
= \nabla_{u}^{T} d G^{-T} J^{T} J G^{-1} \nabla_{ud} = \nabla_{u}^{T} d G^{-1} \nabla_{ud}
\]
Fast marching on parametric surfaces

- Solve eikonal equation in parametrization domain

\[ \nabla_{\vec{u}}^T d G^{-1} \nabla_{\vec{u}} d = 1 \quad d(u_0) = 0 \]

- March on \textit{discretized} parametrization domain.

- We need to express \textbf{update step} in parametrization coordinates.
Fast marching on parametric surfaces

- **Cartesian** sampling of $U$ with unit step.
- Some **connectivity** (e.g. 4- or 8-neighbor).
- Vertex $u_3$ updated from triangle $(u_1, u_2, u_3)$
  \[
  u_1 = u_1 + m_1 = (u_3^{1} + m_1^{1}, u_3^{2} + m_1^{2}) \\
  u_2 = u_2 + m_2 = (u_3^{1} + m_2^{1}, u_3^{2} + m_2^{2})
  \]
- Assuming w.l.o.g. $x_3 = 0$
  \[
  x_1 \approx m_1^{1} r_1 + m_1^{2} r_2 = J m_1 \\
  x_2 \approx m_2^{1} r_1 + m_2^{2} r_2 = J m_2
  \]
  
  or in matrix form
  \[
  V \approx JM
  \]
Fast marching on parametric surfaces

- **Inner product** matrix
  
  \[ E = V^T V \approx M^T G M = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle \end{pmatrix} \]

- Describes triangle geometry.
  - \( \sqrt{e_{ii}} \) lengths of the edges.
  - \( \frac{e_{12}}{\sqrt{e_{11}e_{22}}} \) cosine of the angle.

- Substitute \( Q = E^{-1} \) into the update **quadratic equation**

- Only **first fundamental form coefficients** and **grid connectivity** are required for update.

- Can measure distances when only surface **gradients** are known.
“Unfolding” on parametric surfaces

- **Virtual connections** can be made directly in parametrization domain.

Kimmel & Spira, “An efficient solution to the eikonal equation on parametric manifolds”, 2004
**Heap-based grid update**

- Fast marching and Dijkstra’s algorithm use **heap-based** grid update.
- Next vertex to be updated is decided by extracting the smallest $d$.
- Update order is **unknown** and **data-dependent**.
- Inefficient use of **memory system** and **cache**.
- Inherently **sequential** algorithm – next update depends on previous one.

**Can we do better?**

- **Regular access** to memory (known in advance).
- **Vectorizable** (parallelizable) algorithm.
Marching even faster

- **Danielsson’s algorithm**: update the grid in a **raster scan order**.
- In **Euclidean** case, parametrization is trivial.
- Geodesics are **straight lines** in parametrization domain.
- Each raster scan covers ¼ of the possible directions of the geodesics.
- **Euclidean** distance map computed by four alternating **raster scans**.
Raster scan fast marching

- Generally, geodesics are **curved** in parametrization domain.
- Raster scans have to be **repeated** to produce a convergent solution.
- **Iterative algorithm.**
- Number of iterations **depends** on geometry and parametrization.
- Practically, **few iterations** are required.
Raster scan fast marching

- **What we lost:**
  - No more a **one-pass** algorithm.
  - Computational **complexity** is **data-dependent**.

- **What we found:**
  - **Coherent** memory access, efficient use of **cache**.
  - **No heap**, each iteration is $O(N)$.
  - Raster scans can be **parallelized**.

BBK, “Parallel algorithms for approximation of distance maps on parametric surfaces”, 2007
Parallellization

- **Rotate** scan directions by $45^0$.
- All updates performed along a **row** or **column** can be **parallelized**.
- Constant **CPU** load – suitable for **SIMD** architecture and **GPUs**.
Parallel marching

- **Rotate** scan directions by 45°.
- All updates performed along a **row** or **column** can be **parallelized**.
- Constant **CPU** load.
- Suitable for **SIMD** architecture and **GPUs**.

- **GPU implementation** computes geodesic on grid with 10,000,000 vertices in less than 50 msec.
- About **200 million distances per second**!
Minimal geodesics

- We have a numerical tool to compute *geodesic distance*.
- Sometimes, the *shortest path* itself is needed.

**Minimal geodesics** are *characteristics* of the eikonal equation.

*In other words:*

- Along *geodesic*, eikonal equation becomes an ODE

\[ \dot{\Gamma}(t) = \nabla \chi d(\Gamma(t)) \]

with initial condition \( \Gamma(0) = x_0 \).

- Solve the ODE for \( \Gamma \).
Minimal geodesics

- To find a minimal geodesic between two points \( x_0, x_1 \in X \):
  - Compute \textbf{distance map} from \( x_0 \) to all other points.
  - Starting at \( x_1 \), follow the direction of \(-\nabla_x d\) until \( x_0 \) is reached.
- \textbf{Steepest descent} on the distance map.

\textit{In the parametrization coordinates}

- Let \( \gamma \) be the \textbf{preimage} of \( \Gamma \) in \( U \):
  \[\Gamma(t) = x(\gamma(t))\]
  \[\dot{\Gamma}(t) = J\dot{\gamma}(t)\]
Minimal geodesics

- Substitute into characteristic equation

\[\dot{\gamma} = \nabla_X d(\Gamma)\]
\[J\dot{\gamma} = JG^{-1}\nabla_u d\]
\[\dot{\gamma} = G^{-1}\nabla_u d\]

- Steepest descent on surface = scaled steepest descent in parametrization domain.
Uses of fast marching

- Geodesic distances
- Minimal geodesics
- Voronoi tessellation & sampling
- Offset curves
Implicit surfaces

- Shape represented as level set $X = \{ \psi = 0 \}$ of some $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$
- Examples: medical images, shape-from-X reconstruction, etc.

- Triangulation is costly and potentially inaccurate
Implicit surfaces

- Two-manifold \( X \subset \mathbb{R}^3 \)
- Co-dimension 1

- **Narrow band** of radius \( h \)

\[
B_h(X) = \bigcup_{x \in X} B_{h}^{\mathbb{R}^3}(x)
\]

- Three-manifold with boundary
- \( \partial B_h(X) \) smooth if

\[
h < \frac{1}{\max \kappa_2}
\]
Distances on implicit surfaces

Since $X \subseteq B_h(X)$, for all $x, x' \in X$

$$d_{B_h(X)}(x, x') \leq d_X(x, x') \leq \text{diam } X$$

Similarly, for $h' \leq h$

$$B_{h'}(X) \subseteq B_h(X) \text{ and hence } d_{B_h(X)} \leq d_{B_{h'}(X)}$$

The sequence $\{d_{B_h(X)}\}_h$ is bounded and nondecreasing and hence converges to the supremum of its range $d_{B_h(X)}|_{X \times X} \rightarrow d_X$

For every $\epsilon > 0$ and $x, x' \in X$ there exists such $h > 0$ that

$$|d_{B_h(X)}(x, x') - d_X(x, x')| \leq \epsilon$$
Distances on implicit surfaces

- **Uniform convergence of geodesic distances:**

  For every $\epsilon > 0$ there exists an $h > 0$ such that for every $x, x' \in X$, $|d_{B_h(X)}(x, x') - d_X(x, x')| \leq \epsilon$

- If $\partial X = \emptyset$ then $\max_{x, x' \in X} \|d_{B_h(X)}(x, x') - d_X(x, x')\| \leq C_X \sqrt{h}$

  where $C_X$ is a constant dependent on the geometry of $X$

- **Convergence of minimal geodesics:**

  Let $\Gamma_X : [0, 1] \rightarrow X$ be a unique minimal geodesic between $x, x' \in X$ and let $\Gamma_{B_h(X)} : [0, 1] \rightarrow B_h(X)$ be a minimal geodesic on $B_h(X)$. Then $\max_{t \in [0, 1]} \|\Gamma_X(t) - \Gamma_{B_h(X)}\| \xrightarrow{h \rightarrow 0} 0$

Memoli & Sapiro, “Fast computation of weighted distance functions and geodesics on implicit hyper-surfaces”, 2001
Eikonal equation on implicit surfaces

\[ d : X \times X \rightarrow \mathbb{R} \]

- Explicit
- **Intrinsic** eikonal equation
  \[
  \| \nabla_X d \| = 1
  \\
  d(x_0) = 0
  \]

\[ d : \mathcal{B}_h(X) \times \mathcal{B}_h(X) \rightarrow \mathbb{R} \]

- Implicit
- **Extrinsic** eikonal equation
  \[
  \| \nabla d \| = 1
  \\
  d(x_0) = 0
  \]

**VISCOOSITY SOLUTIONS CONVERGE AS** \( h \rightarrow 0 \)
Narrow band fast marching

- **Euclidean** fast marching on **Cartesian grid**
- Only vertices inside **narrow band** do not participate in update
- Initial values of source set **interpolated** on the grid
- **Heap** or **raster scan** grid visiting