Differential geometry I

Lecture 1
A topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$ (topological disc) is called an $n$-dimensional (or $n$-) manifold.

**Manifolds**

Earth is an example of a 2-manifold.
Charts and atlases

A homeomorphism $\alpha : U_\alpha \to \mathbb{R}^n$ from a neighborhood $U_\alpha$ of $x \in X$ to $\mathbb{R}^n$ is called a **chart**.

A collection of charts whose domains cover the manifold is called an **atlas**.
Numerical geometry of non-rigid shapes

Differential geometry I

Charts and atlases
Smooth manifolds

Given two charts $\alpha : U_\alpha \rightarrow \mathbb{R}^n$ and $\beta : U_\beta \rightarrow \mathbb{R}^n$ with overlapping domains $U_\alpha \cap U_\beta$ change of coordinates is done by transition function

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^n$$

If all transition functions are $C^r$, the manifold is said to be $C^r$

A $C^\infty$ manifold is called smooth
Manifolds with boundary

A topological space in which every point has an open neighborhood homeomorphic to either

- topological disc $\mathbb{R}^n$; or
- topological half-disc $[0, \infty) \times \mathbb{R}^{n-1}$

is called a **manifold with boundary**

Points with disc-like neighborhood are called **interior**, denoted by $\text{Int}(X)$

Points with half-disc-like neighborhood are called **boundary**, denoted by $\partial X$
Embedded surfaces

- Boundaries of tangible physical objects are two-dimensional manifolds.
- They reside in (are embedded into, are subspaces of) the ambient three-dimensional Euclidean space.
- Such manifolds are called embedded surfaces (or simply surfaces).
- Can often be described by the map $x : U \subset \mathbb{R}^2 \to X \subset \mathbb{R}^3$
  - $U \subset \mathbb{R}^2$ is a parametrization domain.
  - the map $x(u) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$ is a global parametrization (embedding) of $X$.
- Smooth global parametrization does not always exist or is easy to find.
- Sometimes it is more convenient to work with multiple charts.
Parametrization of the Earth

\[ U = \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [-\pi, \pi] \]

\[
x^1 = r \cos u^2 \cos u^1 \\
x^2 = r \sin u^2 \cos u^1 \\
x^3 = r \sin u^1
\]
Tangent plane & normal

- At each point $u \in U$, we define a local system of coordinates

  $$x_1 = \frac{\partial x}{\partial u^1} \quad x_2 = \frac{\partial x}{\partial u^2}$$

- A parametrization is regular if $x_1$ and $x_2$ are linearly independent.

- The plane $T_xX = \text{span}\{x_1, x_2\}$ is the tangent plane at $x = x(u)$.

- Local Euclidean approximation of the surface.

- $N \perp T_xX$ is the normal to the surface.
Orientability

- Normal is defined up to a **sign**.
- Partitions ambient space into **inside** and **outside**.
- A surface is **orientable**, if normal $N$ depends smoothly on $x$. 

August Ferdinand Möbius (1800-1868)

Felix Christian Klein (1849-1925)
First fundamental form

- **Infinitesimal displacement** on the chart $du$.

- Displaces $x$ on the surface by

  $$dx = x(u + du) - x(u)$$
  $$= x_1 du^1 + x_2 du^2$$
  $$= J du$$

- $J$ is the **Jacobian matrix**, whose columns are $x_1$ and $x_2$. 
First fundamental form

- **Length** of the displacement
  \[
  d\ell^2 = \|dx\|^2 = du^T J^T J du = du^T G du
  \]

- \( G \) is a **symmetric positive definite** 2×2 matrix.

- Elements of \( G \) are **inner products**
  \[
  g_{ij} = \langle x_i, x_j \rangle
  \]

- Quadratic form
  \[
  d\ell^2 = du^T G du
  \]
  is the **first fundamental form**.
First fundamental form of the Earth

- Parametrization

\[ x = (r \cos u^2 \cos u^1, r \sin u^2 \cos u^1, r \sin u^1) \]

- Jacobian

\[ x_1 = (-r \cos u^2 \sin u^1, -r \sin u^2 \sin u^1, r \cos u^1) \]
\[ x_2 = (-r \sin u^2 \cos u^1, r \cos u^2 \cos u^1, 0) \]

- First fundamental form

\[ G = \begin{pmatrix}
\langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\
\langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle 
\end{pmatrix} \]
\[ = r \begin{pmatrix}
1 & 0 \\
0 & \cos^2 u^1 
\end{pmatrix} \]
First fundamental form of the Earth

\[ G = r \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \]
First fundamental form

- Smooth curve on the chart: \( \gamma : [a, b] \rightarrow U \)
- Its image on the surface: \( \Gamma = x \circ \gamma \)
- Displacement on the curve: \( d\gamma = (\gamma(t + dt) - \gamma(t)) = \dot{\gamma}(t) dt \)
- Displacement in the chart: \( \Gamma = \gamma(t) \)
- Length of displacement on the surface: \( dl = \sqrt{\gamma(t) \cdot T \cdot G(\gamma(t)) \cdot \gamma(t)} dt \)
Intrinsic geometry

- **Length** of the curve

\[ L(\Gamma) = \int_{\Gamma} dl = \int_{a}^{b} \sqrt{\gamma(t)^T G(\gamma(t)) \gamma(t)} \, dt \]

- First fundamental form induces a **length metric (intrinsic metric)**

\[ d_X(x_1, x_2) = \min_{\Gamma} L(\Gamma) \quad \text{with} \quad \Gamma(0)=x_1, \Gamma(1)=x_2 \]

- **Intrinsic geometry** of the shape is **completely described** by the first fundamental form.

- First fundamental form is **invariant to isometries**.
Area

- **Differential area element** on the chart: rectangle $du^1 \times du^2$

- Copied by $x$ to a parallelogram $du^1 x_1 \times du^2 x_2$ in tangent space.

- Differential area element on the surface:

  $$
  da = \|du^1 x_1 \times du^2 x_2\| = \|x_1 \times x_2\| du^1 du^2 = \sqrt{\|x_1\|^2 \|x_2\|^2 - \langle x_1, x_2 \rangle^2} du^1 du^2
  $$

  $$
  = \sqrt{g_{11}g_{22} - g_{12}^2} du^1 du^2
  $$

  $$
  = \sqrt{\det G} du^1 du^2
  $$
Area

- **Area** or a region $\Omega \subseteq X$ charted as $\Omega = x(\omega \subseteq U)$

\[
\mu(\Omega) = \int_\Omega da = \int_\omega \sqrt{\det G} du^1 du^2
\]

- **Relative area**

\[
\nu(\Omega) = \frac{\mu(\Omega)}{\mu(X)}
\]

- **Probability** of a point on $X$ picked at **random** (with uniform distribution) to fall into $\Omega$.

Formally

- $\mu(\Omega), \nu(\Omega)$ are **measures** on $X$. 
Curvature in a plane

- Let $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ be a smooth curve parameterized by arclength
  \[ \int_a^b \| \dot{\gamma}(t) \| \, dt = |a - b| \]

- $\Gamma$ trajectory of a race car driving at constant velocity.
- $\dot{\gamma}$ velocity vector (rate of change of position), tangent to path.
- $\ddot{\gamma}$ acceleration (curvature) vector, perpendicular to path.
- $\kappa = \| \ddot{\gamma} \|_2$ curvature, measuring rate of rotation of velocity vector.
Curvature on surface

- Now the car drives on terrain $X$.
- Trajectory described by $\Gamma : [0, L] \rightarrow X$.
- **Curvature vector** $\vec{\Gamma}$ decomposes into:
  - $P_{T\Gamma X}\vec{\Gamma}$ geodesic curvature vector.
  - $P_N\vec{\Gamma}$ normal curvature vector.
- **Normal curvature** $\kappa_n = \langle N, \vec{\Gamma} \rangle$
- Curves passing in different directions have different values of $\kappa_n$.

*Said differently:*

- A point $x \in X$ has **multiple curvatures**!
Principal curvatures

For each direction \( v \in T_x X \), a curve \( \Gamma \) passing through \( \Gamma(0) = x \) in the direction \( \dot{\Gamma}(0) = v \) may have a different normal curvature \( \kappa_n \).

**Principal curvatures**

\[
\kappa_1 = \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle \\
\kappa_2 = \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle
\]

**Principal directions**

\[
T_1 = \arg \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle \\
T_2 = \arg \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle
\]
Curvature

- **Sign of normal curvature** = direction of rotation of normal to surface.
  - $\kappa_n > 0$ a step in direction $\mathbf{\hat{\Gamma}}$ rotates $\mathbf{N}$ in **same direction**.
  - $\kappa_n < 0$ a step in direction $\mathbf{\hat{\Gamma}}$ rotates $\mathbf{N}$ in **opposite direction**.
Curvature: a different view

- A plane has a constant normal vector, e.g. \( N = (0, 0, 1) \).
- We want to quantify how a curved surface is different from a plane.
- Rate of change of \( N \) i.e., how fast the normal rotates.

- **Directional derivative** of \( N \) at point \( x \in X \) in the direction \( v \in T_x X \)

\[
D_vN = \lim_{t \to 0} \frac{1}{t}(N(\Gamma(t)) - N(x)) = \left. \frac{d}{dt}N(\Gamma(t)) \right|_{t=0}
\]

\( \Gamma : (-\epsilon, +\epsilon) \to X \) is an arbitrary smooth curve with \( \Gamma(0) = x \)
and \( \Gamma'(0) = v \).
Curvature

- $D_vN$ is a vector in $\mathbb{R}^3$ measuring the change in $N$ as we make differential steps in the direction $v$.
- Differentiate $1 = \langle N, N \rangle$ w.r.t. $t$

\[
0 = \frac{d}{dt} \langle N, N \rangle = 2 \langle D_vN, N \rangle
\]

- Hence $D_vN \perp N$ or $D_vN \in T_xX$.
- **Shape operator** (a.k.a. **Weingarten map**): is the map $S : T_xX \to T_xX$ defined by

\[
S(v) = -D_vN
\]
Shape operator

- Can be expressed in **parametrization coordinates** as $S(v) = Sv$

  $S$ is a $2 \times 2$ matrix satisfying

  \[
  \begin{pmatrix}
  S(x_1) \\
  S(x_2)
  \end{pmatrix} = S
  \begin{pmatrix}
  x_1 \\
  x_2
  \end{pmatrix}
  \]

- Multiply by $(x_1, x_2)$

  \[
  \begin{pmatrix}
  S(x_1) \\
  S(x_2)
  \end{pmatrix}(x_1, x_2) = S
  \begin{pmatrix}
  x_1 \\
  x_2
  \end{pmatrix}(x_1, x_2)
  \]

  \[B = SG\]

  where

  \[
  B = \begin{pmatrix}
  \langle S(x_1), x_1 \rangle & \langle S(x_1), x_2 \rangle \\
  \langle S(x_2), x_1 \rangle & \langle S(x_2), x_2 \rangle
  \end{pmatrix} = -\begin{pmatrix}
  \langle \partial_u N, x_1 \rangle & \langle \partial_u N, x_2 \rangle \\
  \langle \partial_u N, x_1 \rangle & \langle \partial_u N, x_2 \rangle
  \end{pmatrix}
  \]
Second fundamental form

- The matrix $B$ gives rise to the **quadratic form**

$$B(v, w) = \langle S(v), w \rangle = w^\top Bv$$

called the **second fundamental form**.

- Related to **shape operator** and **first fundamental form** by identity

$$S = BG^{-1}$$
Principal curvatures encore

- Let $\gamma : [0, L] \to X$ be a curve on the surface.
- Since $\dot{\gamma} \in T_xX$, $\langle \dot{\gamma}, N \rangle = 0$.
- Differentiate w.r.t. to $t$

$$0 = \frac{d}{dt} \langle \dot{\gamma}, N \rangle = \langle \ddot{\gamma}, N \rangle + \langle \dot{\gamma}, \frac{d}{dt}N \rangle$$

$$\kappa_n = \langle \ddot{\gamma}, N \rangle = \langle \ddot{\gamma}, -D_\dot{\gamma}N \rangle = B(\ddot{\gamma}, \dot{\gamma}) = \dot{\gamma}^T B \dot{\gamma}$$

- $\kappa_1 \leq \dot{\gamma}^T B \dot{\gamma} \leq \kappa_2$
- $\kappa_1$ is the smallest eigenvalue of $B$.
- $\kappa_2$ is the largest eigenvalue of $B$.
- $T_1, T_2$ are the corresponding eigenvectors.
Second fundamental form of the Earth

- **Parametrization** \( x = (r \cos u^2 \cos u^1, r \sin u^2 \cos u^1, r \sin u^1) \)

- **Normal**

  \[
  N = (\cos u^2 \cos u^1, \sin u^2 \cos u^1, \sin u^1) \\
  \partial_{u^1} N = (-\cos u^2 \sin u^1, -\sin u^2 \sin u^1, \cos u^1) \\
  \partial_{u^2} N = (-\sin u^2 \cos u^1, \cos u^2 \cos u^1, 0)
  \]

- **Second fundamental form**

  \[
  B = -\left( \begin{array}{cc}
  \langle \partial_{u^1} N, x_1 \rangle & \langle \partial_{u^1} N, x_2 \rangle \\
  \langle \partial_{u^2} N, x_1 \rangle & \langle \partial_{u^2} N, x_2 \rangle 
  \end{array} \right) = -\frac{1}{r} G = \left( \begin{array}{cc}
  -1 & 0 \\
  0 & -\cos^2 u^1
  \end{array} \right)
  \]
Shape operator of the Earth

- First fundamental form

\[ G = r \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \]

- Second fundamental form

\[ B = \begin{pmatrix} -1 & 0 \\ 0 & -\cos^2 u^1 \end{pmatrix} \]

- Shape operator

\[ S = BG^{-1} = -\frac{1}{r} I \]

- Constant at every point.

Is there connection between \textbf{algebraic invariants} of shape operator \( S \) (trace, determinant) with \textbf{geometric invariants} of the shape?
Mean and Gaussian curvatures

- Mean curvature \( H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\text{trace } S \)
- Gaussian curvature \( K = \kappa_1\kappa_2 = \det S \)

hyperbolic point \( K < 0 \)  
elliptic point \( K > 0 \)
Extrinsic & intrinsic geometry

- **First fundamental form** describes completely the **intrinsic geometry**.
- **Second fundamental form** describes completely the **extrinsic geometry** – the “layout” of the shape in ambient space.
- **First fundamental form** is invariant to **isometry**.
- **Second fundamental form** is invariant to **rigid motion (congruence)**.
- If $X$ and $f(X)$ are **congruent** (i.e., $f \in \text{Iso}(\mathbb{R}^3)$), then they have identical intrinsic and extrinsic geometries.
- **Fundamental theorem**: a map preserving the first and the second fundamental forms is a congruence.

*Said differently*: an isometry preserving second fundamental form is a restriction of Euclidean isometry.