Fast marching methods
The continuous way

Metric discretization

Approach I: discrete metric
Discretized shape → Discrete metric → Metrization error → "Sampling theorem" → "Sampling theorem"

Approach II: consistently discretized metric

Discretized metric

Forest fire
- Fire starts at a source $s_0$ at $t = 0$.
- Propagates with constant velocity $c = 1/\alpha$.
- Arrives at time $t(x)$ to a point $x$.
- Fermat’s (least action) principle: The fire chooses the quickest path to travel.
- Governs refraction laws in optics (Snell’s law) and acoustics.
- Fire arrival time $t(x) = \text{distance map } d(x)$ from source.

Distance maps on surfaces
- Distance map on surface $d : X \to \mathbb{R}$
  \[ d(x) = d_X(z_0, x) \]
- Mapped locally to the tangent space
  $d : Tz_0 \to \mathbb{R}$
- A small step in the direction $v \in Tz_0$ changes the distance by
  \[ d(x + v) - d(x) = D_v d(x) + O(||v||^2) \]
- $D_v$ is directional derivative in the direction $v$.

Intrinsic gradient
- For some direction $v_1 \in Tz_0$, $D_v d(x) = d(x)$
- The perpendicular direction $v_2 \perp v_1$ is the direction of steepest change of the distance map.
- $v_2$ is referred to as the intrinsic gradient.
- Formally, the intrinsic gradient of function $d : X \to \mathbb{R}$ at a point $z \in X$ is a map $\nabla_x d : TzX \to TzX$
  satisfying for any $v \in TzX$
  \[ (\nabla_x d)(v) = D_v d(x) \]
Extrinsic gradient
- Consider the distance map as a function \( d : \mathbb{R}^3 \rightarrow \mathbb{R} \).
- The extrinsic gradient of \( d \) at a point \( x \in \mathcal{X} \) is a map \( \nabla_{dX} : \mathcal{X} \rightarrow \mathbb{R}^3 \) satisfying for any direction \( dx \)
  \[
  \langle \nabla_{dX}(x), dx \rangle = \frac{d}{dt} d(x + tdx) \big|_{t=0}
  \]
- In the standard Euclidean basis
  \[
  \nabla_{dX} = \left( \frac{\partial d}{\partial x_1}, \frac{\partial d}{\partial x_2}, \frac{\partial d}{\partial x_3} \right)^T
  \]
- Usually called "the gradient" of \( d \).
- What is the connection between intrinsic and extrinsic gradients?

Eikonal equation
- Let \( \Gamma \) be a minimal geodesic between \( z_0 \) and \( z \).
- The derivative
  \[\dot{\Gamma}(t) = \frac{d}{dt} \Gamma(t)\]
  is the fire front propagation direction.
- In arclength parametrization \( |\dot{\Gamma}(t)|_2 = 1 \).
- Fermat’s principle:
  \[\dot{\Gamma}(t) = \nabla_{dX}(d(\Gamma(t)))\]
- Propagation direction = direction of steepest increase of \( d \).
- Geodesic is perpendicular to the level sets of \( d \) on \( \mathcal{X} \).

Uniqueness of solution
- In classic PDE theory, a solution is a continuous differentiable function \( d : \mathcal{X} \rightarrow \mathbb{R} \) satisfying
  \[
  |\nabla_{dX}(d(x))|_2 = 1 \\
  d(z_0) = 0
  \]
- PDE theory guarantees existence and uniqueness of solution.
- Distance map is not everywhere differentiable.
- Solution is not unique!
Sub- and super-derivatives (1D case)

\[ D^+ d = \{ |a| \leq 1 \} \]
\[ D^- d = \{ |a| \leq 1 \} \]

- Superderivative: the set of all slopes above the graph
  \[ D^+ d(x) = \left\{ a : \lim_{h \to 0^+} \frac{d(x+h) - d(x) - ah}{h} \leq 0 \right\} \]
- Subderivative: the set of all slopes below the graph
  \[ D^- d(x) = \left\{ a : \lim_{h \to 0^-} \frac{d(x+h) - d(x) - ah}{h} \geq 0 \right\} \]
- \[ D^+ d = D^- d = \{ d \} \] where \( d \) is differentiable.

Fast marching methods (FMM)

- A family of numerical methods for solving eikonal equation.
- Finds the viscosity solution = distance map.
- Simulates wavefront propagation from a source set.
- A continuous variant of Dijkstra’s algorithm.
- Consistently approximate the intrinsic metric on the surface.

Dijkstra’s algorithm

- Initialize \( d(x_0) = 0 \) and \( d(x) = \infty \) for other vertices and mark them as green.
- Initialize queue of red vertices \( Q = \emptyset \).
- While \( Q \neq \emptyset \):
  - Find vertex with smallest value of \( d \).
  - \( x = \arg \min_{x' \in Q} d(x') \)
  - For each unprocessed adjacent vertex \( x' \in X(x) \cap \{ Q \}
    \[ d(x') = \min \{ d(x'), d(x) + \ell(x,x') \} \]
  - Remove \( x \) from \( Q \)
  - Return distance map \( d(x) = d_{C}(x_0, x) \).

Update step

- Dijkstra’s update
  - Vertex \( x \) updated from adjacent vertex \( x_1 \)
  - Distance \( d(x) \) computed from \( d(x_1) \)
  - Path restricted to graph edges
- Fast marching update
  - Vertex \( x \) updated from adjacent triangle \( (x_1, x_2, x_3) \)
  - Distance \( d(x) \) computed from \( d(x_1) \) and \( d(x_2) \)
  - Path can pass on mesh faces
Fast marching update step
- Update $x_3$ from triangle $(x_1, x_2, x_3)$
- Compute $d(x_3)$ from $d_1 = d(x_1)$ and $d_2 = d(x_2)$
- Model wave front propagating from planar source
  $$(x, n) + p = 0$$
- $n$ unit propagation direction
- $p$ source offset
- Front hits $x_1$ at time $d_1$
- Hits $x_2$ at time $d_2$
- When does the front arrive to $x_3$?

Fast marching update step
- Assume w.l.o.g. $x_1, x_2, x_3 \in \mathbb{R}^2$ and $x_3 = 0$.
- $d_3$ is given by the point-to-plane distance
  $$d_3 = \langle x_1, n \rangle$$
- Solve for parameters $n$ and $p$ using the point-to-plane distance
  $$(x_1, n) + p = d_1$$
  $$(x_2, n) + p = d_2$$
- In vector notation
  $$V^T n + p \cdot 1 = d$$
  where $V = (x_1, x_2)$. $d = (d_1, d_2)^T$, and $1 = (1, 1)^T$.
- In a non-degenerate triangle matrix $V$ is full-rank
  $$n = (V^T)^{-1} (d - p \cdot 1) = V^{-1} (d - p \cdot 1)$$

Causality condition
- Quadratic equation is satisfied by both $n$ and $-n$.
- Two solutions for $d_3$.
- Causality: front can propagate only forward in time.
- Causality condition
  $$d_3 > d_1, d_2$$
  $$d_3 \cdot 1 > V^T n + p \cdot 1$$
  $$d_3 \cdot 1 > V^T n + d_1, d_2 \cdot 1$$
  $$0 > V^T n$$

Monotonicity condition
- Viscosity solution has to be a monotonically increasing function.
- Monotonicity condition: $d_3$ increase when $d_1$ or $d_2$ increase.
  In other words:
  $$\nabla_d d_3 = \begin{pmatrix} \frac{\partial d_3}{\partial d_1} \\ \frac{\partial d_3}{\partial d_2} \end{pmatrix}^T > 0$$
- Differentiate
  $$d_3 \cdot 1^T Q_1 - 2d_3 \cdot 1^T Q_d + d^T Q d - 1 = 0$$
  w.r.t $d = (d_1, d_2)^T$ obtaining
  $$\nabla_d d_3 = \frac{Q(d - d_3 \cdot 1)}{1^T Q (d - d_3 \cdot 1)}$$
Monotonicity condition

\[ \nabla_d d_3 = \frac{Q(d - d_3)}{1^T Q(d - d_3) \cdot 1} \]

- Substitute \( n = V^{-T} (d - d_3) \cdot 1 \)

\[ \nabla_d d_3 = \frac{Q V^T n}{1^T Q V^T n} \]

- Monotonicity \( \nabla_d d_3 > 0 \) satisfied when both coordinates of \( Q V^T n \) have the same sign.
- \( Q = (V^T V)^{-1} \) is positive definite
- Causality condition: \( V^T n < 0 \)
- Monotonicity condition: \( Q V^T n < 0 \)

Monotonicity condition

- Since \( Q = (V^T V)^{-1} \) we have \( Q V^T V = I \)
- Rows of \( Q V^T \) are orthogonal to triangle edges

Geometric interpretation:

- \( n \) must come from within the triangle.
- Said differently:

Fast marching update

- Solve the quadratic equation and select the largest solution

\[ d_3^2 \cdot 1^T Q_1 - 2 d_3 \cdot 1^T Q d + d^T Q d - 1 = 0 \]

- Compute propagation direction

\[ n = V^{-T} (d - d_3 \cdot 1) \]

- If monotonicity condition \( Q V^T n < 0 \) is violated,

\[ d_3 = \min \{ d_1 + ||x_1 - x_3||_2, d_2 + ||x_2 - x_3||_2 \} \]

- Set

\[ d(x_3) = \min \{ d(x_3), d_3 \} \]

Fast marching on obtuse meshes

- Inconsistent solution if the mesh contains obtuse triangles
- Remeshing is costly
- Solution: split obtuse triangles by adding virtual connections to non-adjacent vertices
- Done as a pre-processing step in \( O(N) \)
Mesh “unfolding”

- Virtual connection splits obtuse angle into two acute ones.

Kimmel & Sethian, “Computing geodesic paths on manifolds”, 1998

MATLAB® intermezzo

Fast marching

Eikonal equation on parametric surfaces

- Parametrization \( x: U \rightarrow \mathbb{R}^2 \) of \( X \) over \( U \subset \mathbb{R}^2 \).
- Compute distance map \( d: U \rightarrow \mathbb{R} \), \( d(\mathbf{s}) = d(x(x(u,v),v)) \) from source \( a_0 \in U \).
- Chain rule
  \[
  \frac{\partial d}{\partial u} = \frac{\partial d}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial d}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial d}{\partial x} \frac{\partial x}{\partial v}
  \]
- Extrinsic gradient in parametrization coordinates
  \( \nabla_{x^d} = J^T \nabla d \)
- Intrinsic gradient in parametrization coordinates
  \( \nabla_{x^d} = J ( J^T J^{-1} )^T \nabla d = J G^{-1} \nabla d \)

Fast marching on parametric surfaces

- Solve eikonal equation in parametrization domain
  \[
  \nabla_{x^d} d G^{-1} \nabla d = 1 \quad d(a_0) = 0
  \]
- March on discretized parametrization domain.
- We need to express update step in parametrization coordinates.
Fast marching on parametric surfaces

- Cartesian sampling of $U$ with unit step.
- Some connectivity (e.g. 4- or 8-neighbor).
- Vertex $v_0$ updated from triangle $(v_{11}, v_{12}, v_{13})$
  
  $v_1 = v_0 + v_{11}$
  
  $v_2 = v_0 + v_{12}$
  
  Assuming w.l.o.g. $v_3 = 0$
  
  $v_3 = v_{13} + m_i v_{12} = J_{v_3}$
  
  $v_4 = v_{13} + m_j v_{11} = J_{v_4}$

or in matrix form

$V = JM$

Fast marching Methods

Spira & Kimmel, “An efficient solution to the eikonal equation on parametric manifolds”, 2004

Fast marching on parametric surfaces

- Inner product matrix
  
  $B = V^T V = \sum_{l \geq 2} \left( v_l, v_l \right)$

- Describes triangle geometry.
  
  $\sqrt{e_{ij}}$ lengths of the edges.
  
  $e_{12}/\sqrt{e_{11} e_{22}}$ cosine of the angle.

- Substitute $Q = E^{-1}$ into the update quadratic equation
  
  $d_0^2 \cdot E^{-1} d_1 - 2 d_0 \cdot E^{-1} d_2 + d_0^2 E^{-1} d_2 - 1 = 0$

- Only first fundamental form coefficients and grid connectivity are required for update.

- Can measure distances when only surface gradients are known.

Unfolding on parametric surfaces

- Virtual connections can be made directly in parametrization domain.

Heap-based grid update

- Fast marching and Dijkstra’s algorithm use heap-based grid update.

- Next vertex to be updated is decided by extracting the smallest $d$.

- Update order is unknown and data-dependent.

- Inefficient use of memory system and cache.

- Inherently sequential algorithm – next update depends on previous one.

- Can we do better?
  
  - Regular access to memory (known in advance).
  
  - Vectorizable (parallelizable) algorithm.

Marching even faster

- Danielsson’s algorithm: update the grid in a raster scan order

- In Euclidean case, parametrization is trivial.

- Geodesics are straight lines in parametrization domain.

- Each raster scan covers ¼ of the possible directions of the geodesics.

- Euclidean distance map computed by four alternating raster scans.

Raster scan fast marching

- Generally, geodesics are curved in parametrization domain.

- Raster scans have to be repeated to produce a convergent solution.

- Iterative algorithm.

- Number of iterations depends on geometry and parametrization.

- Practically, few iterations are required.
**MATLAB® intermezzo**

**Raster scan fast marching**

What we lost:
- No more a one-pass algorithm.
- Computational complexity is data-dependent.

What we found:
- Coherent memory access, efficient use of cache.
- No heap, each iteration is $O(N)$.
- Raster scans can be parallelized.

BBK, “Parallel algorithms for approximation of distance maps on parametric surfaces”, 2007

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**Parallelization**

- Rotate scan directions by $45^\circ$.
- All updates performed along a row or column can be parallelized.
- Constant CPU load – suitable for SIMD architecture and GPUs.

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**Minimal geodesics**

- We have a numerical tool to compute geodesic distance.
- Sometimes, the shortest path itself is needed.

**Minimal geodesics** are characteristics of the eikonal equation.

In other words:
- Along geodesic, eikonal equation becomes an ODE
  
  $\dot{\gamma}(t) = \nabla_x \phi(\gamma(t))$

  with initial condition $\gamma(0) = x_0$.
- Solve the ODE for $\Gamma$.

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**Parallel marching**

- Rotate scan directions by $45^\circ$.
- All updates performed along a row or column can be parallelized.
- Constant CPU load.
- Suitable for SIMD architecture and GPUs.

GPU implementation computes geodesic on grid with 10,000,000 vertices in less than 50 msec.
- About 200 million distances per second!
Processing and Analysis of Geometric Shapes  Fast Marching Methods

Minimal geodesics
- Substitute into characteristic equation
  \[
  \dot{r} = \nabla d_r(x(t)) \\
  \dot{y} = J\dot{r}^{-1}v_N \\
  \dot{y} = \sigma^{-1}v_N
  \]
- Steepest descent on surface = scaled steepest descent in parametrization domain.

Uses of fast marching
- Geodesic distances
- Minimal geodesics
- Voronoi tessellation & sampling
- Offset curves

Implicit surfaces
- Shape represented as level set \( \mathbf{X} = \{ \psi = 0 \} \) of some \( \psi : \mathbb{R}^3 \rightarrow \mathbb{R} \)
- Examples: medical images, shape-from-X reconstruction, etc.
- Triangulation is costly and potentially inaccurate

Distances on implicit surfaces
- Since \( \mathbf{X} \subseteq \partial \Omega(X) \) for all \( m,n \in \mathbf{X} \)
  \( d_{\partial \Omega(X)}(m,n) \leq d_{\partial \Omega(X)}(m,n) \leq \text{diam } \mathbf{X} \)
- Similarly, for \( \mathbf{X} \subseteq \mathbf{X} \)
  \( d_{\mathbf{X}}(m,n) \leq d_{\mathbf{X}}(m,n) \)
- The sequence \( \{d_{\partial \Omega(X)}(m,n)\}_k \) is bounded and nondecreasing and hence converges to the supremum of its range \( d_{\partial \Omega(X)}(m,n) \)
- For every \( \epsilon > 0 \) and \( m,n \in \mathbf{X} \) there exists such \( \delta > 0 \) that \( d_{\partial \Omega(X)}(m,n) - d_{\partial \Omega(X)}(m,n) \leq \epsilon \)

Memoli & Sapiro, “Fast computation of weighted distance functions and geodesics on implicit hyper-surfaces”, 2001
Eikonal equation on implicit surfaces

\[ d : X \times X \rightarrow \mathbb{R} \]

- Explicit
- Intrinsic eikonal equation

\[ \nabla d(x, y) = 1 \]
\[ d(m_0) = 0 \]

Eikonal equation on explicit surfaces

\[ d : X \rightarrow \mathbb{R} \]

- Implicit
- Extrinsic eikonal equation

\[ \nabla d(x) = 1 \]
\[ d(x_0) = 0 \]

VISCOSITY SOLUTIONS CONVERGE AS \( \epsilon \rightarrow 0 \)

Narrow band fast marching

- Euclidean fast marching on Cartesian grid
- Only vertices inside narrow band do not participate in update
- Initial values of source set interpolated on the grid
- Heap or raster scan grid visiting