**Introduction to geometry**

The German way

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**Manifolds**

A topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$ (topological disc) is called an $n$-dimensional (or $n$-) manifold.

**Charts and atlases**

A homeomorphism $\alpha : U_\alpha \to \mathbb{R}^n$ from a neighborhood $U_\alpha$ of $x \in \mathcal{X}$ to $\mathbb{R}^n$ is called a chart. A collection of charts whose domains cover the manifold is called an atlas.

**Smooth manifolds**

Given two charts $\alpha : U_\alpha \to \mathbb{R}^n$ and $\beta : U_\beta \to \mathbb{R}^n$ with overlapping domains $U_\alpha \cap U_\beta$ change of coordinates is done by transition function $\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$.

If all transition functions are $C^r$, the manifold is said to be $C^r$. A $C^\infty$ manifold is called smooth.

**Manifolds with boundary**

A topological space in which every point has an open neighborhood homeomorphic to either:
- topological disc $\mathbb{R}^n$, or
- topological half-disc $[0, \infty) \times \mathbb{R}^{n-1}$

is called a manifold with boundary.

Points with disc-like neighborhood are called interior, denoted by $\text{int}(\mathcal{X})$.

Points with half-disc-like neighborhood are called boundary, denoted by $\partial \mathcal{X}$.
Embedded surfaces

- Boundaries of tangible physical objects are two-dimensional manifolds.
- They reside in (are embedded into, are subspaces of) the ambient three-dimensional Euclidean space.
- Such manifolds are called embedded surfaces (or simply surfaces).
- Can often be described by the map \( \mathbf{x} : U \subset \mathbb{R}^2 \to X \subset \mathbb{R}^3 \)
  - \( U \subset \mathbb{R}^2 \) is a parameterization domain.
  - the map \( \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \)
    is a global parameterization (embedding) of \( X \).
- Smooth global parameterization does not always exist or is easy to find.
- Sometimes it is more convenient to work with multiple charts.

Tangent plane & normal

- At each point \( \mathbf{x} \in U \), we define local system of coordinates
  \[ \mathbf{s}_1 = \frac{\partial \mathbf{x}}{\partial u} \quad \mathbf{s}_2 = \frac{\partial \mathbf{x}}{\partial v} \]
- A parametrization is regular if \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) are linearly independent.
- The plane \( T_{\mathbf{x}} X = \text{span}(\mathbf{s}_1, \mathbf{s}_2) \)
  is tangent plane at \( \mathbf{x} = \mathbf{x}(u, v) \).
- Local Euclidean approximation of the surface.
- \( N \perp T_{\mathbf{x}} X \) is the normal to surface.

First fundamental form

- Infinitesimal displacement on the chart \( ds \)
- Displaces \( \mathbf{x} \) on the surface by
  \[ ds = \mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v) = s_1 du + s_2 dv \]
- \( J \) is the Jacobean matrix, whose columns are \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \).

Parametrization of the Earth

\[ U = \left[ \frac{\mathbf{w}}{\sqrt{2}} \right] \times [-\pi, \pi] \]
\[ s^2 = r \cos^2 \theta \cos^2 \phi \]
\[ s^3 = r \sin^2 \theta \cos^2 \phi \]
\[ s^1 = r \sin \theta \cos \phi \]

Orientability

- Normal is defined up to a sign.
- Partitions ambient space into inside and outside.
- A surface is orientable, if normal \( \mathbf{N} \) depends smoothly on \( \mathbf{x} \).

Möbius stripe
Klein bottle
(3D section)

First fundamental form

- Length of the displacement
  \[ ds^2 = |ds|^2 = ds^T J^T J ds \]
- \( G \) is a symmetric positive definite 2x2 matrix.
- Elements of \( G \) are inner products
  \[ g_{ij} = (s_i, s_j) \]
- Quadratic form
  \[ ds^2 = ds^T G ds \]
  is the first fundamental form.
First fundamental form of the Earth

- Parametrization
  \[ z = (r \cos u \cos v, r \sin u \cos v, r \sin v) \]
- Jacobian
  \[ x_1 = (-r \cos u \sin v, r \sin u \cos v, r \cos v) \]
  \[ x_2 = (-r \sin u \cos v, r \cos u \sin v, r \sin v) \]
- First fundamental form
  \[ g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \cos^2 u & 0 \\ 0 & 0 & r^2 \sin^2 u \cos^2 v \end{pmatrix} \]

Intrinsic geometry

- Length of the curve
  \[ \ell(\gamma) = \int_{\gamma} ds = \int_{0}^{1} \sqrt{g(\gamma'(t)) \cdot \gamma'(t)} \, dt \]
- First fundamental form induces a length metric (intrinsic metric)
  \[ ds^2 = g_{ij}(x_1, x_2) \, dx^i \, dx^j \]
- Intrinsic geometry of the shape is completely described by the first fundamental form.
- First fundamental form is invariant to isometries.

Area

- Differential area element on the chart: rectangle \( dx_1 \times dx_2 \)
- Copied by \( \gamma \) to a parallelogram \( d\alpha_1 \times d\alpha_2 \) in tangent space.
- Differential area element on the surface:
  \[ d\alpha = \sqrt{|e_1^2 e_2 e_3^2 - e_1 e_2 e_3 (e_1 e_2 e_3)|} \, ds \]

Area

- Area or a region \( \Omega \subset X \) charted as \( \Omega = \alpha(\omega \subset \mathbb{R}^2) \)
  \[ x(\omega) = \int_{\omega} ds = \int_{\omega} \sqrt{g(\gamma'(t)) \cdot \gamma'(t)} \, dt \]
- Relative area
  \[ \sigma(\alpha) = \frac{\mu(\alpha)}{\mu(X)} \]
- Probability of a point on \( X \) picked at random (with uniform distribution) to fall into \( \Omega \).
  Formally
  \[ p(\Omega) \sigma(\Omega) \] are measures on \( X \).
Curvature in a plane
- Let \( \gamma : [a, b] \to \mathbb{R}^2 \) be a smooth curve parameterized by arclength
  \[ \int_a^b ||\gamma'(t)|| dt = |a - b| \]
- \( \gamma' \) trajectory of a race car driving at constant velocity.
- \( \gamma'' \) velocity vector (rate of change of position), tangent to path.
- \( \gamma''' \) acceleration (curvature) vector, perpendicular to path.
- \( k = ||\gamma'''|| \) curvature, measuring rate of rotation of velocity vector.

Curvature on surface
- Now the car drives on terrain \( X \).
- Trajectory described by \( \gamma : [a, b] \to X \).
- Curvature vector \( \gamma' \) decomposes into
  - \( P_{\gamma'} \) geodesic curvature vector.
  - \( P_{\gamma''} \) normal curvature vector.
- Normal curvature \( n_\gamma = \langle X, \gamma' \rangle \)
- Curves passing in different directions have different values of \( n_\gamma \).
  Said differently:
  - A point \( \gamma \in X \) has multiple curvatures!

Principal curvatures
- For each direction \( v \in T_\gamma X \), a curve \( \gamma \) passing through \( \gamma(0) = \gamma \) in the direction \( \gamma'(0) = v \) may have a different normal curvature \( \kappa_v \).
- Principal curvatures \( \kappa_1 = \max_{v \in T_\gamma X} \kappa_v \) \( \kappa_2 = \min_{v \in T_\gamma X} \kappa_v \)
- Principal directions \( v_1 = \arg \max_{v \in T_\gamma X} \kappa_v \) \( v_2 = \arg \min_{v \in T_\gamma X} \kappa_v \)

Curvature a different view
- A plane has a constant normal vector, e.g. \( X = (0, 0, 1) \).
- We want to quantify how a curved surface is different from a plane.
- Rate of change of \( X \), i.e., how fast the normal rotates.
- Directional derivative of \( X \) at point \( \gamma \in X \) in the direction \( v \in T_\gamma X \)
  \[ D_v X = \lim_{h \to 0} \frac{1}{h} (X(\gamma(0)) - X(\gamma(h))) = \frac{d}{dt} X(\gamma(t)) \bigg|_{t=0} \]
- \( \gamma : [a, b] \to X \) is an arbitrary smooth curve with \( \gamma(0) = \gamma \) and \( \gamma'(0) = v \).

Curvature
- \( D_v N \) is a vector in \( T_{\gamma'} X \) measuring the change in \( N \) as we make differential steps in the direction \( v \).
- Differentiate \( L = \langle N, N \rangle \) w.r.t. \( t \)
  \[ \begin{align*}
    0 &= \frac{d}{dt} \langle N, N \rangle \\
    &= 2 \langle DN, N \rangle
  \end{align*} \]
- Hence \( DN \cdot LN = DN \cdot LN \in T_{\gamma'} X \).
- Shape operator (a.k.a. Weingarten map): is the map \( S : T_{\gamma'} X \to T_{\gamma'} X \) defined by
  \[ S(v) = -DN \]
Shape operator

- Can be expressed in parametrization coordinates as $G(u) = du$
- $G$ is a 2x2 matrix satisfying
  $$\begin{pmatrix} G(u_1) \\ G(u_2) \end{pmatrix} = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
- Multiply by $(u_1, u_2)$
  $$\begin{pmatrix} G(u_1) \\ G(u_2) \end{pmatrix} (u_1, u_2) = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (u_1, u_2)$$
  $$B = BG$$

Second fundamental form

- The matrix $B$ gives rise to the quadratic form
  $$B(v, w) = B(v, w) = \psi^T B \psi$$
- Called the second fundamental form.
- Related to shape operator and first fundamental form by identity
  $$B = BG^{-1}$$

Principal curvatures encore

- Let $\gamma : I \to \mathbb{R}^3$ be a curve on the surface.
- Since $\gamma' \times \gamma'' \neq 0$
- Differentiate w.r.t. $t$
  $$\frac{d}{dt} (\gamma', \gamma'', \gamma''') = (\gamma' \times \gamma'')$$
- Normal
  $$n = (\gamma', \gamma', \gamma'') = (\gamma', -D_t N) = B(t', t'') = t'' B^{-1}$$
- $n_1 \leq t'' B^{-1} \leq n_2$
- $n_1$ is the smallest eigenvalue of $B$.
- $n_2$ is the largest eigenvalue of $B$.
- $\mathcal{H}, \mathcal{E}$ are the corresponding eigenvectors.

Mean and Gaussian curvatures

- Mean curvature
  $$H = \frac{1}{2} (n_1 + n_2) = \frac{1}{2} \text{trace} B$$
- Gaussian curvature
  $$K = n_1 n_2 = \text{det} B$$

- Hyperbolic point $K < 0$
- Elliptic point $K > 0$
Extrinsic & intrinsic geometry
- First fundamental form describes completely the intrinsic geometry.
- Second fundamental form describes completely the extrinsic geometry – the "layout" of the shape in ambient space.
- First fundamental form is invariant to isometry.
- Second fundamental form is invariant to rigid motion (congruence).
- If \( X \) and \( f(X) \) are congruent (i.e., \( f \in \text{Isom}(\mathbb{R}^3) \) ), then they have identical intrinsic and extrinsic geometries.
- Fundamental theorem: a map preserving the first and the second fundamental forms is a congruence. Said differently: an isometry preserving second fundamental form is a restriction of Euclidean isometry.

Riemannian geometry
- Riemannian metric: bilinear symmetric positive definite smooth map \( g_x : T_x X \times T_x X \to \mathbb{R} \)
- Abstract inner product on tangent space of an abstract manifold.
- Coordinate-free.
- In parametrization coordinates is expressed as first fundamental form.
- A farewell to extrinsic geometry!

An intrinsic view
- Our definition of intrinsic geometry (first fundamental form) relied so far on ambient space.
- Can we think of our surface as an abstract manifold immersed nowhere?
- What ingredients do we really need?
  - Smooth two-dimensional manifold
  - Tangent space \( T_x X \) at each point.
  - Inner product \( \langle \cdot, \cdot \rangle_x : T_x X \times T_x X \to \mathbb{R} \)
- These ingredients do not require any ambient space!

Nash’s embedding theorem
- Embedding theorem (Nash, 1956): any Riemannian metric can be realized as an embedded surface in Euclidean space of sufficiently high yet finite dimension.
- Technical conditions:
  - Manifold is \( C^4 \text{, } n \geq 3 \)
  - For an \( m \)-dimensional manifold, embedding space dimension is \( n = m^2 + m + 3 \)
- Practically: intrinsic and extrinsic views are equivalent!

Uniqueness of the embedding
- Nash’s theorem guarantees existence of embedding.
- It does not guarantee uniqueness.
- Embedding is clearly defined up to a congruence.
- Are there cases of non-trivial non-uniqueness?
  - Formally:
    - Given an abstract Riemannian manifold \((S, g)\), and an embedding \( \varepsilon : S \to \mathbb{R}^3 \), does there exist another embedding \( \gamma : S \to \mathbb{R}^3 \) such that \( X = \varepsilon(U) \) and \( Y = \gamma(U) \) are incongruent?
  - Said differently:
    - Do isometric yet incongruent shapes exist?
Processing & Analysis of Geometric Shapes

**Bending**

- Shapes admitting incongruent isometries are called **bendable**.
- Plane is the simplest example of a bendable surface.
- **Bending**: an isometric deformation transforming $X$ into $Y$.

**Bending and rigidity**

- Existence of two incongruent isometries $X, Y$ does not guarantee that $X$ can be physically folded into $Y$ without the need to cut or glue.
- If there exists a family of bendings $f_t$ continuous w.r.t. $t$ such that $f_0(X) = X$ and $f_1(Y) = Y$, the shapes are called **continuously bendable** or applicable.
- Shapes that do not have incongruent isometries are **rigid**.
- **Extrinsic geometry** of a rigid shape is fully determined by the intrinsic one.

**Alice’s wonders in the Flatland**

- Subsets of the plane: $X \subset \mathbb{R}^2$
- Second fundamental form vanishes everywhere
- Isometric shapes $X$ and $Y$ have identical first and second fundamental forms
- Fundamental theorem: $X$ and $Y$ are congruent.

**Rigidity conjecture**

- If the faces of a polyhedron were made of metal plates and the polyhedron edges were replaced by hinges, the polyhedron would be rigid.
- In practical applications shapes are represented as polyhedra (triangular meshes), so…

**Rigidity conjecture timeline**

- 1766: Euler’s Rigidity Conjecture: every polyhedron is rigid
- 1813: Cauchy: every convex polyhedron is rigid
- 1927: Cohn-Vossen: all surfaces with positive Gaussian curvature are rigid
- 1974: Gluck: almost all simply connected surfaces are rigid
- 1977: Connelly finally disproves Euler’s conjecture

**Connelly sphere**

- Isocahedron
  - Rigid polyhedron
- Connelly sphere
  - Non-rigid polyhedron
“Almost rigidity”
- Most of the shapes (especially, polyhedra) are rigid.
- This may give the impression that the world is more rigid than non-rigid.
- This is probably true, if isometry is considered in the strict sense
  \[ \frac{\partial L(f(x), f(x'))}{\partial x} = \frac{\partial L(x, x')}{\partial x} \]
- Many objects have some elasticity and therefore can bend almost isometrically
  \[ \frac{\partial L(f(x), f(x'))}{\partial x} \approx \frac{\partial L(x, x')}{\partial x} \]
- No known results about “almost rigidity” of shapes.

Gaussian curvature – a second look
- Gaussian curvature measures how a shape is different from a plane.
- We have seen two definitions so far:
  - Product of principal curvatures: \( K = R_1 R_2 \)
  - Determinant of shape operator: \( K = \det S \)
- Both definitions are extrinsic.

Here is another one:
- For a sufficiently small \( r \), perimeter of a metric ball of radius \( r \) is given by
  \[ P(r) = 2\pi r - \frac{\pi}{3} r^3 + O(r^4) \]

Gaussian curvature – a second look
- Riemannian metric is locally Euclidean up to second order.
- Third order error is controlled by Gaussian curvature.
- Gaussian curvature
  \[ K = \lim_{\varepsilon \to 0} \frac{2\pi r - P(r)}{\varepsilon^3} \]
- \( 2\pi r - P(r) \) measures the defect of the perimeter, i.e., how \( P(r) \) is different from the Euclidean \( 2\pi r \).
  - positively curved surface – perimeter smaller than Euclidean.
  - negatively curved surface – perimeter larger than Euclidean.

Theorema egregium
- Our new definition of Gaussian curvature is intrinsic!
- Gauss’ Remarkable Theorem
  \[ \text{...formula itaque sponte perducit ad egregium theorema: si superficies curva in quamcunque aliam superficiem explicat, mensura curvarum in singulis punctis invariata manet.} \]
  \[ \text{In modern words:} \]
  - Gaussian curvature is invariant to isometry.

An Italian connection...
Gauss-Bonnet formula

- Solution: integrate Gaussian curvature over the whole shape
  \[ \chi_X = \int_X K \, dA \]
- \( \chi_X \) is Euler characteristic.
- Related genus by
  \[ \chi_X = 2 - 2 \text{genus } X \]
- Stronger topological rather than geometric invariance.
- Result known as Gauss-Bonnet formula.

Intrinsic Invariants

- We all have the same Euler characteristic \( \chi = 2 \).
- Too crude a descriptor to discriminate between shapes.
- We need more powerful tools.