Problems
Numerical Geometry
Numerical geometry of non-rigid shapes

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1. Show that if in a compact metric space there exists an $r/3$-covering containing $n$ points, then an $r$-separated set cannot contain more than $n$ points.

2. Show that in a compact metric space, a maximal $r$-separated set is an $r$-covering.

3. Prove that the farthest point strategy produces an $r$-separated $r$-covering.

4. Prove that a convex set is homeomorphic to a disk.

5. Prove that centroidal Voronoi tessellation minimizes the variance of the representation error.

6. Show an example of a geodesic triangle, whose circumscribing ball is non-unique due to insufficient sampling density.

7. Show an example when the Delaunay tessellation of a surface does not exist due to insufficient sampling density.

8. Show an example when the Delaunay tessellation is not unique.

9. Show that given a smooth compact surface $X$ embedded into $\mathbb{R}^3$, there exists an open set $U_X$ such that $X \subseteq U_X$, and a continuous map $\xi : U_X \to X$, such that for all $u \in U_X$, the point $\xi(u)$ is the orthogonal projection of $u$ onto $X$ and it is unique.

10. Show that $|\rho(x) - \rho(x')| \leq d_{\mathbb{R}^3}(x, x')$ for all $x, x' \in X$. 
11. Show the relation

\[ \rho(x) \leq \frac{1}{\max\{\kappa_1(x), \kappa_2(x)\}} \]

between the local feature size \( \rho \) and the maximum curvature radius.

12. Prove that the area of a triangle with vertices \( x_1, x_2, x_3 \in \mathbb{R}^3 \) can be expressed as \( \frac{1}{2} \| (x_2 - x_1) \wedge (x_3 - x_1) \|_2 \).

13. Validate the Schwarz lantern example by a formal proof.