Problems

Numerical Optimization

Numerical geometry of non-rigid shapes

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1. Describe a function satisfying the optimality condition (O'1) and a weaker version of (O'2): $\nabla^2 f(x^*) \geq 0$. Is it guaranteed that x^* is a local minimum?

2. Show that matrix multiplication is commutative under the trace operator, i.e., $\text{trace}(A^{\text{T}}B) = \text{trace}(BA^{\text{T}})$ for A and B of size $n \times m$.

3. Verify the matrix functions gradients $\nabla\text{trace}(AX) = A^{\text{T}}$ and $\nabla\text{trace}(X^{\text{T}}BX) = (B + B^{\text{T}})X$ by coordinate-wise differentiation in the standard basis.

4. Show that a matrix is positive-semidefinite if and only if its eigenvalues are non-negative.

5. Show that an intersection of convex sets is a convex set.

6. Show that a function $f : A \subseteq V \to \mathbb{R}$ is convex if and only if its epigraph $\text{epi}f = \{(v, y) \in A \times \mathbb{R} : f(v) \leq y\}$ is a convex set.

7. Show that if a function $f : A \subseteq V \to \mathbb{R}$ is convex, its sub-level sets $\{v \in A : f(v) \leq y\}$ are convex sets.

8. Show that a local minimum of a convex function is (i) a global minimum, and (ii) it is unique if the function is strictly convex.

9. Show that Armijo rule eventually terminates.

10. Generalize the gradient descent example to a general positive-definite quadratic function in $\mathbb{R}^n$. Derive the descent direction and the optimal step size.

11. Prove the chain rule $\nabla_x f(Ax) = A \nabla_x f(Ax)$.

12. Derive the expression

$$d_{\text{nsd}} = -\left(\nabla f(x)^{\mathrm{T}} Q^{-1} \nabla f(x)\right)^{-1/2} Q^{-1} \nabla f(x).$$

13. Derive the Newton iteration as the best possible preconditioning.

14. Derive the Broyden approximation for the Hessian by solving

$$\begin{aligned} H^{(k+1)}(x^{(k+1)} - x^{(k)}) &= \nabla f(x^{(k+1)}) - \nabla f(x^{(k)}), \\ H^{(k+1)} u &= H^{(k)} u \quad \text{s.t.} \quad \langle x^{(k+1)} - x^{(k)}, u \rangle = 0. \end{aligned}$$

Show that $H^{(k+1)}$ defined in this way is unique.

15. Find the Lagrange multipliers in the following constrained problem,

$$\min_{(x^1,x^2)^{\mathrm{T}} \in \mathbb{R}^2} x^1 + x^2 \quad \text{s.t.} \quad \begin{cases} (x^1 - 1)^2 + (x^2)^2 = 1 \\ (x^1 - 2)^2 + (x^2)^2 = 4. \end{cases}$$

Does this result contradict the KKT conditions? Explain.

16. Find the solution of the constrained optimization problem

$$\min_{(x^1,x^2)^{\mathrm{T}} \in \mathbb{R}^2} x^1 + x^2 \quad \text{s.t.} \quad (x^1)^2 + (x^2)^2 = 2,$$

using geometric interpretation of KKT conditions only.