Problems

Spectral Embedding

Numerical geometry of non-rigid shapes

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1. Let XX^{T} be a Gram matrix, where X is an $N \times m$ matrix. Prove the following properties:

   (a) $XX^{\mathrm{T}} \succeq 0$,

   (b) $\mathrm{rank}(XX^{\mathrm{T}}) = \mathrm{rank}(X)$.

2. Show that the classic MDS solution minimizes the strain $\sigma_{\mathrm{F}}(Z)$.

3. Show that one way to obtain the matrix K_X is

$$K_X = -\frac{1}{2} J (D_X \odot D_X) J,$$

   where $\odot$ denotes the Hadamard (element-wise) matrix product, such that $(D_X \odot D_X)_{ij} = d_X^2(x_i, x_j)$ and $J = \left(I_N - N^{-1} 1_{N\times N}\right)$ is the *centering matrix*. Extend this result by replacing J by a general projection matrix $P = I_N - 1_{N\times N} w^{\mathrm{T}}$, where w is a vector satisfying $w^{\mathrm{T}} 1_{N\times 1} = 1$.

4. Establish the duality between classic MDS and PCA.

5. Derive the expression

$$\sigma_{\mathrm{LOC}}(Z) \quad = \quad \mathrm{trace}(Z^{\mathrm{T}} L_X Z).$$

6. Show that the Laplacian matrix L_X is positive semidefinite.

7. Show that the minimum eigenvalue problem $Ax = \lambda_{\min} x$, where $A \succeq 0$, is equivalent to the constrained optimization problem

$$\min_{x \in \mathbb{R}^N} x^{\mathrm{T}} A x \quad \text{s.t.} \quad x^{\mathrm{T}} x = 1.$$

Extend this result by showing that

$$\min_{X \in \mathbb{R}^{N \times m}} \operatorname{trace}(X^{\mathrm{T}} A X) \quad \text{s.t.} \quad X^{\mathrm{T}} X = I,$$

is equivalent to finding the m smallest eigenvalues of A.

8. Show that ∇f projected onto $T_x X$ coincides with $\nabla_X f$.