$$R(\xi) = \int_{U_X} \|\nabla \xi\|^2_{\mathrm{HS}} \sqrt{\det G_X} du, \tag{12.8}$$

is independent of the choice of parameterization, and can be rewritten in terms of $\varphi : X \to Y$ as

$$R(\varphi) = \int_X \|\nabla_X \varphi\|^2_Y \, da. \tag{12.9}$$

All quantities are now intrinsic, meaning that $R(\varphi)$ is also invariant to isometries of X and Y.

Litke *et al.* [254] give a physical interpretation to this regularizer. In elasticity theory, the term $G_X^{-1}(\nabla\xi)^{\mathrm{T}}(G_Y \circ \xi)\nabla\xi$ is known as the *Cauchy-Green deformation tensor*, which expresses the square of local changes in distances due to an elastic deformation. Its trace, expressed by $\|\nabla\xi\|^2_{\mathrm{HS}}$, measures the square of the average local change of length due to the deformation created by pressing a thin rubber shell X into a mold having the form of Y. R itself is called the *Dirichlet energy* functional and measures (up to a factor of $\frac{1}{2}$) the elastic energy of the deformation.

12.3 Minimum distortion correspondence

The choice of an intrinsic local match error, like the one based on the Gaussian curvature, and an intrinsic regularizer like (12.9), make the minimization problem (12.6) invariant to isometries. By combining the two terms of the minimized functional, we can formulate a general isometry-invariant correspondence problem as

$$\min_{\varphi : X \to Y} \int_X e(x, \varphi) \, da,$$

where $e(s, \varphi)$ can be expressed as

$$e(x, \varphi) = \int_X \sigma(d_X(x, x'), d_Y(\varphi(x), \varphi(x'))) \, da,$$

with $\sigma(d_X, d_Y)$ depending only on the metrics d_X and d_Y. Consequently, our correspondence problem assumes the form

$$\min_{\varphi : X \to Y} \int_{X \times X} \sigma(d_X(x, x'), d_Y(\varphi(x), \varphi(x'))) \, da \times da. \tag{12.10}$$

This minimization problem can be recognized as a GMDS problem with some generalized distortion expressed as the integral over $X \times X$, in which σ can be interpreted as a local stress. For the particular choice of $\sigma(d_X, d_Y) = (d_X - d_Y)^2$, the familiar least-squares version of GMDS is obtained.

Figure 12.1. A minimum distortion correspondence between different postures of a male figure computed using GMDS. Corresponding regions on the meshes are denoted with the same colors (see insert for image in color). Note that the correspondence is defined up to a self-isometry (symmetry) of both shapes; for example, the correspondence between the two left-most and the two right-most shapes is reflected.

If previously we were looking for the distortion dis φ, which quantified the intrinsic dissimilarity of the two objects, we are now interested in φ itself, or formally,

$$\varphi = \arg \min_{\varphi : X \to Y} \text{dis } \varphi. \tag{12.11}$$

We call this map the *minimum-distortion correspondence* between X and Y. An example is shown in Figure 12.1. Note that the computation of φ can be performed either directly on the surfaces themselves or in the parameterization space if X and Y admit a global parameterization. The map φ can also be interpreted as a parameterization of Y in X.

Minimum distortion correspondence is not limited to isometric surfaces. Many classes of objects though not isometric share common (intrinsic) geometric properties – for example, two different faces have nose, eyes, mouth, and so forth. Finding the minimum distortion correspondence between two faces can be visualized as the problem of putting a flexible rubber mask over one face, trying to minimize its stretch. Obviously, in most cases we will place the mask in such a way that the facial features coincide, at least roughly, so that the correspondence is semantically correct. Practice shows that even when dealing with two substantially different objects, the minimum distortion map usually gives a reasonable correspondence (Figure 12.2). The approach can also be applied to partially similar objects using the scheme described in Chapter 11. An example is shown in Figure 12.3.

It is worthwhile noting that any intrinsic correspondence, including the one computed using GMDS, is defined up to the isometry groups of the two

Figure 12.2. A minimum distortion correspondence between male, female, and gorilla figures computed using GMDS. Corresponding regions on the meshes are denoted with the same colors (see insert for image in color). The male-female correspondence (left) appears accurate (up to a symmetry reflecting the left and the right) despite the differences in the intrinsic geometry. On the other hand, the minimum distortion correspondence between the male and the gorilla figures fails, as the human legs are mapped to the elongated gorilla's hands (second from the right). Manually fixing four semantically correct landmarks in GMDS initialization fixes the correspondence (right).

surfaces, meaning that if one or both surfaces have symmetries, there will be multiple mappings $\varphi : X \to Y$ achieving the minimum distortion. In such a case, there is no way to give any preference to one of these correspondences based on the intrinsic geometry only, and we have to introduce extrinsic or other (e.g., photometric) information in order to make the right choice. For example, our face is known to have an approximate reflection symmetry with respect to the vertical axis. Consequently, there may exist two minimum distortion correspondences, one mapping the left eye of X to the left eye of Y, and another mapping the left eye of X to the right eye of Y (Figure 12.1). Adding an extrinsic constraint demanding that the correspondence does not change the surface *orientation* would exclude the second possibility.

12.4 Texture mapping and transfer

Imagine that an animation studio wishes to create a synthetic animation character for its new movie. Developing an animation system that will allow the character's face to reproduce basic human expressions is a Sisyphean work,