2.8 Bending and rigidity

Though the Nash embedding theorem guarantees that any Riemannian metric can be realized as an embedded surface, it says nothing about the uniqueness of such a realization. As a particular case, let us consider a Riemannian manifold $\mathcal{S}$, whose metric is realized in $\mathbb{R}^3$ by the embedding $x : \mathcal{S} \to \mathbb{R}^3$, that is, the intrinsic metric induced on the embedded surface $X = x(\mathcal{S})$ coincides with $d_{\mathcal{S}}$. Clearly, the embedding $x : \mathcal{S} \to \mathbb{R}^3$ is not unique, as for any congruence $h \in \mathrm{Iso}(\mathbb{R}^3)$, $h \circ x(\mathcal{S})$ also realizes $d_{\mathcal{S}}$. This type of non-uniqueness is rather trivial, and we are interested in richer non-uniqueness going beyond Euclidean isometries. The question of existence of a non-unique embedding can be therefore posed as whether $d_{\mathcal{S}}$ can be realized by another embedding $y : \mathcal{S} \to \mathbb{R}^3$, such that $Y = y(\mathcal{S})$ is incongruent with $X = x(\mathcal{S})$. Said differently, we are looking for X and Y realizing the same intrinsic geometry while differing in their extrinsic geometries.

It appears that some surfaces have non-unique embeddings in $\mathbb{R}^3$. The simplest example is the plane, which can be bent and folded in many ways. A more sophisticated example is shown in Figure 2.7. Let us assume that $\mathcal{S}$ admits two embeddings x and y. The map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f = y \circ x^{-1}$ describes the extrinsic deformation that we need to apply to X in order to obtain Y. Such a deformation is called *bending*, and a surface having a non-unique embedding is called *bendable*. Clearly, because $f \circ x(\mathcal{S}) = y(\mathcal{S})$ and X and Y are isometric, f must be distance preserving, that is,

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

for every $x_1, x_2 \in X$. Here, d_X and d_Y denote the intrinsic metrics on the embedded surfaces X and Y, respectively, induced by the Euclidean metric in $\mathbb{R}^3$.

Considering the example in Figure 2.7, it is easy to transform one wine bottle into another by sawing off the bottle neck and welding it back "upside down." Now imagine that instead of glass the bottle is made of flexible, yet inelastic material. Trying to push the neck inside the bottle, we will soon realize that it is an impossible task, although the two versions of the bottle are isometric. That is, there is no way to turn the bottle neck upside down without distorting it – the only way to do so necessarily involves a cut. This experiment brings us to the notion of continuous bending. Two isometric surfaces X and Y are called *applicable* or *continuously bendable* if there exists a family of bendings $\{f_\lambda\}_{\lambda=0}^{1}$ continuous with respect to λ, such that $f_0(X) = X$ and $f_1(X) = Y$. Physically, applicability means that given a surface X realized as a thin shell of inelastic material, it can be pressed without tearing into a mold having the form of Y. Being continuously bendable is a stronger property than being bendable; in fact, the wine bottle example shows that incongruent embeddings of the same surface are not necessarily applicable to each other. A particularly interesting class of continuously bendable surfaces

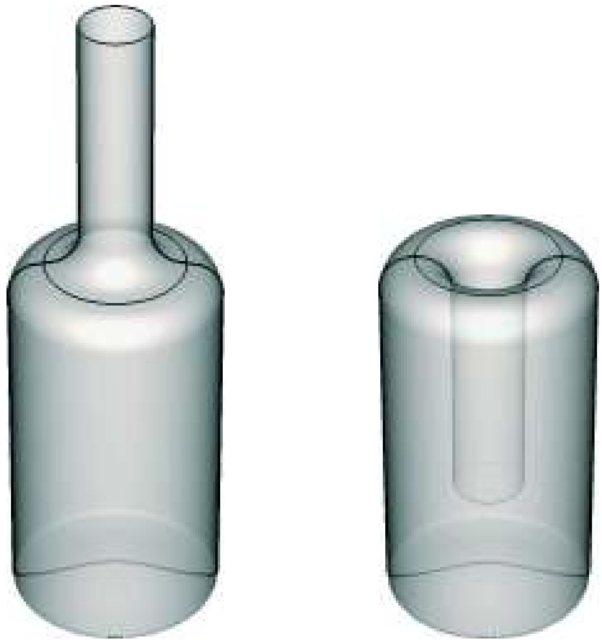

Figure 2.7. An example of non-rigid objects. If we cut the neck of the wine bottle on the left along the indicated circle and weld it back upside down, we will obtain the object on the right. The new bottle is isometric to the original one, yet the two objects are incongruent. Note that while the bottle is bendable, it is not continuously bendable.

are *flat* surfaces, which are applicable to a subset of the plane.[18] Such surfaces can be flattened onto a plane without distortion and thus can be constructed by cutting, folding, and bending a sheet of paper (Figure 2.8). Flat surfaces are important in manufacturing, especially in shipbuilding, where different parts of a ship are created from sheet steel.

Along with bendable and continuously bendable surfaces, many other surfaces admit a unique embedding into $\mathbb{R}^3$ (of course, up to a Euclidean isometry). Such surfaces, whose extrinsic geometry is completely determined by the intrinsic one, are called *non-bendable* or *rigid*.

Example 2.11 (rigidity of planar shapes). Let X and Y be two subsets of the plane with the intrinsic metrics d_X and d_Y, respectively, induced by the Euclidean metric. Because both X and Y are restricted to the plane, their second fundamental forms are identically zero. As a consequence, if X and Y are isometric, they are necessarily congruent. This result implies that planar shapes are rigid, as their geometry is completely determined by the first fundamental form. This fact has an interesting consequence. If $\mathcal{S}$ is a flat surface, one of its embeddings is a planar shape $X = x(\mathcal{S}) \subset \mathbb{R}^2$. This means that the

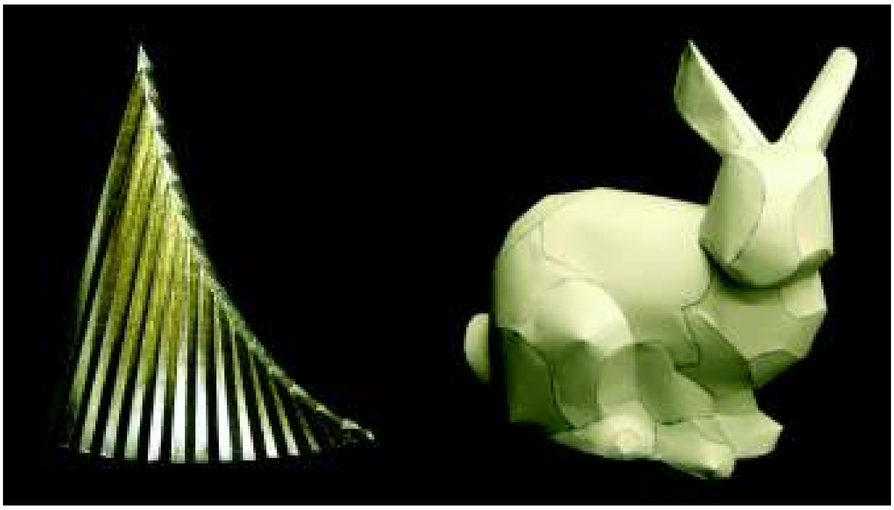

Figure 2.8. Left: origami, the art of paper folding, consists of creating intricate curved shapes from a flat piece of paper (image courtesy Alex Hubris). Right: approximation of the Stanford Bunny as developable patches glued together (reproduced from [354] with permission).

isometry group of $\mathcal{S}$ is isomorphic to the isometry group of X, comprising all distance preserving mappings $h : X \to X$. However, we have seen that X is rigid, which implies that every such h must be a restriction of a Euclidean isometry to X. Hence, any $g \in \mathrm{Iso}(\mathcal{S})$ can be represented as the composition $g = (x|_X)^{-1} \circ h \circ x$, where h is an Euclidean isometry obeying $h(X) = X$. Intuitively, this means that studying the isometry (intrinsic symmetry) group of $\mathcal{S}$ can be replaced by applying the surface to the plane followed by studying the extrinsic symmetries of the obtained planar shape.

The question of rigidity interested many mathematicians for centuries, who focused mostly on the class of *polyhedral* surfaces. Let us briefly review the history of some dramatic developments in the field. As a starting point we should probably consider 1766, when Euler proposed his renowned rigidity conjecture, stating that all closed polyhedra embedded in $\mathbb{R}^3$ are rigid. In 1813, Cauchy (then only 24 years old) proved that *convex* polyhedra are, indeed, rigid [94].[19] In 1974, Gluck showed that almost all triangulated simply connected closed surfaces are rigid, remarking that Euler was right "statistically" [174]. Informally, Gluck's result implies that picking a closed polyhedron "at random," the probability that it will bend is zero. In 1977, Euler's conjecture was finally disproved by Connelly [114], who found a simple bendable closed polyhedron (Figure 2.9, right), sometimes referred to as the *Connelly sphere* [113, 115]. Unlike polyhedra, much less is known about the rigidity of smooth surfaces. One of the main results was proved in 1927 by Cohn-Vossen

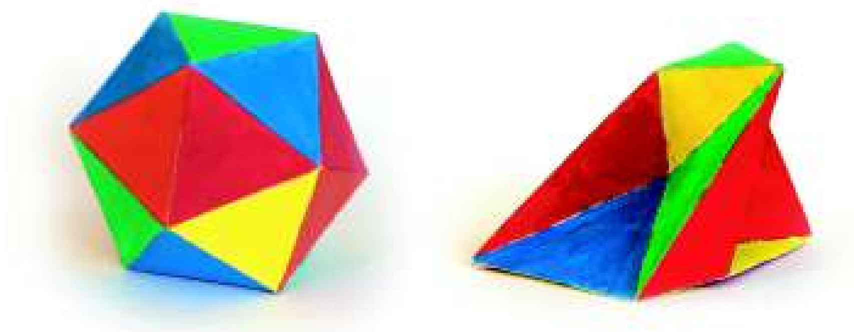

Figure 2.9. A paper realization of a convex polyhedron with twenty faces (left), and a variant of the Connelly sphere (right). Unlike the icosahedron, the Connelly sphere is non-rigid and can be bent along some of its edges.

[108], who showed that all surfaces with strictly positive Gaussian curvature are rigid. Cohn-Vossen's rigidity theorem can be thought of as a continuous analog to Cauchy's theorem for convex polyhedra.

These results may give the impression that most surfaces around us are rigid, which is probably true if rigidity is considered in a strict sense. The situation changes dramatically if small distortions are allowed. Although, to the best of our knowledge, the question of whether or not two rigid surfaces can be bent by an ϵ-isometric bending has not been addressed in the literature, practice shows that a great variety of surfaces appearing rigid in the strict sense, can be bent *almost isometrically*, or at least, can be realized with low distortion by many incongruent surfaces in $\mathbb{R}^3$. In a sense, a significant part of this book is motivated by this astonishing fact.

2.9 Intrinsic invariants

So far, we have seen that the Gaussian curvature of a surface can be defined as a product of the two principal curvatures or the determinant of the shape operator. Apart from these two definitions, there exists yet another one. Imagine an insect tied with a thread of length r to some point x on the surface. The insect makes a round trip around x while the thread is completely stretched on the surface, and we measure the perimeter $P(r)$ of the closed curve it describes. For a flat surface, we would obviously get $P(r) = 2\pi r$. When we repeat the experiment on a curved surface, $P(r)$ will differ from $2\pi r$, and measuring this discrepancy, we hope to realize how our surface is different from a plane. It appears that for a sufficiently small r, we will measure