is called the *Laplace-Beltrami operator*, or sometimes the *Laplacian* for short.[10] The Laplace-Beltrami operator is expressible solely in terms of the Riemannian metric and is therefore an intrinsic property of the shape. It can be alternatively defined as $\Delta_X f = \mathrm{trace}(\nabla^2 f)$, from which it follows that the Laplace-Beltrami operator is the non-Euclidean analog of the Laplacian operator: in the Euclidean case, we have $\Delta f = -(f_{xx} + f_{yy})$.

Using Δ_X, we can rewrite problem (8.10) as

$$\min_{f \in L_2(X)} \int_X f(x)\Delta_X f(x) da \quad \text{s.t.} \quad \|f\|^2_{L_2(X)} = 1. \tag{8.11}$$

Note that now the integral $\int_X f(x)\Delta_X f(x)da$ in (8.11) replaces the term $z^{\mathrm{T}} L_X z$ we had in the discrete case, and the Laplace-Beltrami operator Δ_X takes the role of the Laplacian matrix L_X.

8.4 To hear the shape of the drum

In 1787, the German physicist Ernst Chladni, considered by many the father of acoustics, published the book *Entdeckungen über die Theorie des Klanges* ("Discoveries concerning the theory of sound") [102]. In his book, Chladni described a famous experiment for the visualization of vibrations produced by acoustic waves: covering a thin plate with sand and making it vibrate by running a violin bow. The sand was observed to accumulate in certain regions, producing patterns of beautiful complexity. A modern physicist would call these shapes *stationary waves*. Mathematically, the behavior of such waves is governed by the *stationary Helmholtz equation* (representing the spatial part of the wave equation solutions),[11] which in our notation reads

$$\Delta_X f = \lambda f.$$

It follows that the beautiful patterns observed by Chladni are *eigenfunctions* (the continuous analog of eigenvectors) of the Laplace-Beltrami operator Δ_X.[12] An example of such eigenfunctions is shown in Figure 8.3.

For compact shapes, the spectrum of Δ_X is discrete, that is, there exists a countable set of the solutions to the equation $\Delta_X f = \lambda f$. Because the Laplace-Beltrami operator is an intrinsic property of the shape, its spectrum (the set of all eigenvalues) is isometry-invariant (this property is exemplified in Figure 8.4). Based on this property, Reuter *et al.* proposed an isometry-invariant description of shapes by numerically approximating the Laplace-Beltrami spectrum [323] (a related approach was also described in [330]).

However, does the Laplace-Beltrami spectrum completely characterize the intrinsic geometry of a shape? More formally, we want to know whether two shapes with the same spectrum (such shapes are called *isospectral*) are necessarily isometric. The question whether a Riemannian surface can be determined by its spectrum was asked for the first time in the early 1960s by

Figure 8.3. The first non-trivial eigenfunctions of the Laplace-Beltrami operator of a human shape. Colors and contours visualize the values of the eigenfunctions at each point of the shape.

Leon Green. Because of the acoustic interpretation of the Laplace-Beltrami spectrum, Mark Kac has posed this question metaphorically in a 1966 paper entitled "Can one hear the shape of the drum?" [215]. Kac addressed the particular case of the Laplace-Beltrami spectrum on planar domains and is

Figure 8.4. Empirical evidence of the fact that the Laplace-Beltrami operator is an intrinsic property of a shape. Shown is one of the eigenfunctions of the Laplace-Beltrami operator, which remains approximately without changes despite the near-isometric deformations of the shape.

quoted to have said, "Personally, I believe that one cannot hear the shape, but I may well be wrong and I am not prepared to bet large sums either way."

In Riemannian geometry, it is known that the area of two isospectral surfaces is equal, which implies that the area can be "heard" from the Laplace-Beltrami spectrum [394, 27]. Other "audible" properties include the total Gaussian curvature $\int_X K da$ and Euler characteristic χ [265]. Yet, generally, a Riemannian manifold cannot be determined by its spectrum. In other words, one cannot hear the shape of the drum. This fact is supported by numerous counterexamples of manifolds that are isospectral but not isometric [389, 90, 82, 83, 180].

On the other hand, in order to apply methods based on the Laplace-Beltrami spectra to the problem of non-rigid shape representation, it is crucial to know how different the classes of isospectral and isometric surfaces are. That is, whether there exist large classes of non-isometric but isospectral surfaces, and how different (non-isometric) can isospectral surfaces be. For the time being, this is an open research question.

8.5* Discrete Laplace-Beltrami operator

Before concluding this chapter, there are a few words to say about the discrete approximation of the Laplace-Beltrami operator. The resemblance of the Laplace-Beltrami operator to the graph Laplacian may create a wrong impression that the two are equivalent. First, note the Laplace-Beltrami operator is