Problems

Taste of Geometry

Numerical geometry of non-rigid shapes

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1. Show that a bi-Lipschitz function is injective (i.e. there exists an inverse from its image) and its inverse is Lipschitz continuous.

2. Describe the isometric group for the following metric spaces:

   (a) Equilateral triangle in the plane with the standard Euclidean metric.

   (b) Perfect n-gon in the plane with the standard Euclidean metric.

   (c) A unit two-dimensional sphere $\mathbb{S}^2$ with the intrinsic metric induced from the ambient Euclidean space $\mathbb{R}^3$.

3. Show that

$$d_L(x,y) \quad = \quad \inf_{\gamma} \{L(\gamma) \;\; \text{s.t. } \gamma : [a,b] \to X, \gamma(a) = x_1, \gamma(b) = x_2\}.$$

   stemming from a length structure is a metric.

4. Show that the length structure induced by the intrinsic metric d_L induces the same intrinsic metric d_L.

5. Articulate the difference between isometries and arcwise isometries (maps preserving length structures) by showing examples of arcwise isometries, which are not isometries.

6. Show that in a set A of a metric space (X, d_X), $d_X|_A$ coincides with the intrinsic metric on A if and only if the shortest path between every pair of points in A lies completely inside A.

7. Prove that the quadratic form $d\ell^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2$ is positive definite if and only if the parameterization $x : U \to \mathbb{R}^3$ is regular.
8. Prove that $S(x_i) = -\partial_{u^i}N$.
9. Prove that the second fundamental form is symmetric, i.e., $B(v, w) = B(w, v)$.
10. Prove that the elements of B are also given by $b_{ij} = \langle N, \frac{\partial^2 x}{\partial u^i \partial u^j} \rangle$.
11. Prove that the spectrum of the shape operator is invariant to the choice of coordinates.
12. Express the first and the second fundamental forms of the surface given as the graph of a function, $(x, y, z(x, y))$. Give an expression for the shape operator and the mean and Gaussian curvatures.
13. Prove that a curve with everywhere vanishing geodesic curvature κ_g is a geodesic.
14. Show surfaces with identical second fundamental forms, yet different first fundamental forms.
15. Show that the Gaussian curvature is an intrinsic quantity by expressing the determinant of the shape operator in terms of the first fundamental form and its derivatives.
16. Prove equivalence of the intrinsic and the extrinsic definitions of the Gaussian curvature.
17. The Gaussian curvature has an alternative intrinsic definition: consider a geodesic triangle ABC on the surface, i.e., a triangle composed of the intersection of three geodesics. The angles in the triangles are measured between the vectors in the tangent space at the three vertices. Define the *defect* of the triangle as $\delta(ABC) = \alpha + \beta + \gamma - \pi$, i.e. how the sum of angles in the triangle is different from the $180°$ obtained on the plane. The limit

$$K = \lim_{ABC \searrow x} \frac{\delta(ABC)}{\text{Area}(ABC)}$$

defines the Gaussian curvature at the point x, where $ABC \searrow x$ means that the triangle converges to a point. Prove equivalence of the two intrinsic definitions.